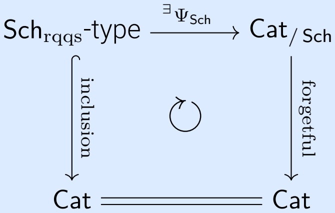
Category-Theoretic Reconstruction

of Rings and Schemes



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Abstract

I will explain an algorithm for reconstructing
the underlying scheme of the objects of $\mathsf{Sch}_{\blacklozenge/S}$, where

- S : a locally Noetherian normal scheme,
- $\blacklozenge/S$: a set of properties of S-schemes s.t.
  $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$, and
- $\mathsf{Sch}_{\blacklozenge/S}$: the full subcategory of $\mathsf{Sch}_{/S}$ determined by
  the objects $X \in \mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property of $\blacklozenge/S$.

To explain this algorithm,
I will talk about a Ring version of this algorithm: $\mathsf{Alg}_{R/\blacklozenge} \rightsquigarrow R$.

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Notations

S, R $\cdots$ a base scheme, a coefficient ring, resp.

$\blacklozenge/S$, $R/\blacklozenge$

$\cdots$ sets of properties of S-schemes and R-algebras, resp.

$\mathsf{Sch}_{\blacklozenge/S}$, $\mathsf{Alg}_{R/\blacklozenge}$

$\cdots$ full subcats $\mathsf{Sch}_{\blacklozenge/S} \subset \mathsf{Sch}_{/S}$ and $\mathsf{Alg}_{R/\blacklozenge} \subset \mathsf{Alg}_{R/}$ det'nd by $\blacklozenge$.

$\times, \lim, \operatorname{colim}$

$\cdots$ fiber prods, limits and colims in Sch, Alg.

$\times^{\blacklozenge}, \lim^{\blacklozenge}, \operatorname{colim}^{\blacklozenge}, \otimes^{\blacklozenge}$

$\cdots$ fiber prods, limits, colims and tensor prods in $\mathsf{Sch}_{\blacklozenge/S}, \mathsf{Alg}_{R/\blacklozenge}$.

We shall mainly be concerned with the properties

$$\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}\,.$$

Research

Mochizuki 2004

$\blacklozenge/S = \text{f.t.}/S$, S: locally Noetherian, $\mathsf{Sch}_{\blacklozenge/S} \rightsquigarrow S$.

Mochizuki 2014

log scheme version of [Mzk04]

van Dobben de Bruyn 2019

$\blacklozenge = \varnothing$, R and S: arbitrary, $\mathsf{Sch}_{/S} \rightsquigarrow S$, $\mathsf{Alg}_{R/} \rightsquigarrow R$.

Anabelian Geometry

$\blacklozenge/S = \text{fét}/S$, S: a variety over an "arithmetic" field, etc...

Remark. [vDdB19] $\not\Rightarrow$ [Mzk04].

Definitions

Definition. We say that a category $\mathcal{C}$ is **of schematic type** if it admits an equivalence of categories $\mathcal{C} \xrightarrow{\sim} \mathsf{Sch}_{\blacklozenge/S}$, where

- S is a locally Noetherian normal scheme, and
- $\blacklozenge \subset \{\mathrm{red}, \mathrm{qsep}, \mathrm{qcpt}, \mathrm{sep}\}$ a set of properties.

Definition. We write $\mathsf{Sch}_{\mathrm{rqqs}}$-type for the 2-groupoid of categories of schematic type, equivalences, and natural isomorphisms.

Definition. Let $\mathcal{C}$ be a category and $c \in \mathcal{C}$ an object.

- We shall write $\mathcal{C}_{/c}$ for the slice category.
- We shall write $\mathcal{C}_{c/}$ for the coslice category.

Main Theorem

The inclusion $\mathsf{Sch}_{\mathrm{rqqs}}$-type $\hookrightarrow$ Cat admits a factorization

$$\mathsf{Sch}_{\mathrm{rqqs}}\text{-type} \xrightarrow{\Psi} \mathsf{Cat}_{/\,\mathsf{Sch}} \xrightarrow{\text{forgetful}} \mathsf{Cat}$$

such that for any locally Noetherian normal scheme S and set of properties $\blacklozenge \subset \{\mathrm{red},\ \mathrm{qsep},\ \mathrm{qcpt},\ \mathrm{sep}\}$,
$\Psi(\mathsf{Sch}_{\blacklozenge/S})$ is isomorphic to the forgetful functor: $\mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}$.

Corollary. $S, T \neq \varnothing$: locally Noetherian normal schemes.
Then, $\mathsf{Sch}_{\blacklozenge/S} \cong \mathsf{Sch}_{\lozenge/T} \ \Rightarrow\ $"$\blacklozenge = \lozenge$".

Corollary. S, T: locally Noetherian normal schemes.
Then, the following functor is an equivalence:

$$\mathrm{Isom}(S,T) \xrightarrow{\sim} \mathbf{Isom}(\mathsf{Sch}_{\blacklozenge/T}, \mathsf{Sch}_{\blacklozenge/S})$$

$$f \mapsto f^* \overset{\text{def}}{:=} S \times^{\blacklozenge}_{f,T} (-).$$

Idea of Proof

To prove the main theorem,
we need to construct a functor $\Psi : \mathsf{Sch}_{\mathrm{rqqs}}$-type $\to \mathsf{Cat}_{/\,\mathsf{Sch}}$.

Hence, $\forall \mathcal{C} \in \mathsf{Sch}_{\mathrm{rqqs}}$-type, we need to construct $\Psi(\mathcal{C}) : \mathcal{C} \to \mathsf{Sch}$.
In particular, we need to construct a functor
$\Psi(\mathsf{Sch}_{\blacklozenge/S}) : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}$
which is isomorphic to the forgetful functor.

> To Do. For any object $X \in \mathsf{Sch}_{\blacklozenge/S}$ and morphism $f \in \mathsf{Sch}_{\blacklozenge/S}$,
> we need to construct a scheme $\tilde{X} \cong X$
> and a morphism of schemes $\tilde{f} \cong f$, functorially,
> from the intrinsic structure of the abstract category $\mathsf{Sch}_{\blacklozenge/S}$.

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A Theorem

To explain an example of a reconstruction algorithm,
we introduce a proof of the folowing theorem:

Theorem. [van Dobben de Bruyn] Let R, S be rings. Then

$$\mathrm{Isom}(R,S) \xrightarrow{\sim} \mathbf{Isom}(\mathsf{Alg}_{R/}, \mathsf{Alg}_{S/}),$$

$$f \mapsto f^* \overset{\text{def}}{:=} S \otimes_{f,R} (-).$$

To prove above theorem, we remember the following well-known result:

Well Known Result. [Beck] A : a ring, M : an A-mod.
Then the square-zero extension $A \ltimes M$ admits an abelian group object structrure in $\mathsf{Alg}_{A/A} \overset{\text{def}}{:=} (\mathsf{Alg}_{A/})_{/A}$, moreover,

$$\mathsf{Mod}_A \xrightarrow{\sim} \mathsf{Ab}(\mathsf{Alg}_{A/A}),$$

$$M \mapsto A \ltimes M.$$

A Theorem

Remark. By considering the well-known isom $R \xrightarrow{\sim} \mathrm{End}(\mathrm{id}_{\mathsf{Mod}_R})$, we obtain

$$\mathsf{Alg}_{R/} \cong \mathsf{Alg}_{S/} \;\Rightarrow\; R \cong S.$$

However, the operator $\mathrm{End}(\mathrm{id}_{\mathsf{Mod}_*})$ is not functorial wrt "*".
Hence we can't conclude the theorem from only this observation.

Lemma. Let $A \in \mathsf{Alg}_{R/}$ be an object, $A_1 = A \ltimes M_1$, $A_2 = A \ltimes M_2$, and $A \to B \xrightarrow{q} A$ be objects of $\mathsf{Ab}(\mathsf{Alg}_{A/A})$.
Then $\ker(q) \cong M_1 \otimes_A M_2$ if and only if
B is "closest" to $A_1 \otimes_A A_2$ among objects satisfying:

- B admits a morphism $B \to A_1 \otimes_A A_2$ in $\mathsf{Alg}_{A/A}$ such that $A \xrightarrow{\sim} B \times_{A_1 \otimes_A A_2} (A_1 \oplus A_2)$.

A Theorem

The above lemma conclude that for any category $\mathcal{C} \cong \mathsf{Alg}_{R/}$ and $X \in \mathcal{C}$, we can define category-theoretically a monoidal structure $\otimes$ on $\mathsf{Ab}(\mathcal{C}_{X/X})$ such that $\mathsf{Ab}(\mathsf{Alg}_{A/A})^{\otimes} \cong \mathsf{Mod}_A^{\otimes}$.

Moreover, we can observe easily that for any $(A \to B) \in \mathsf{Alg}_{R/}$, the functor $\mathsf{Ab}(\mathsf{Alg}_{A/A}) \to \mathsf{Ab}(\mathsf{Alg}_{B/B})$ induced by $B \otimes_A (-)$ coincides with $B \otimes_A (-)$, via $\mathsf{Mod}_A \xrightarrow{\sim} \mathsf{Ab}(\mathsf{Alg}_{A/A})$.

We define $\Psi(\mathcal{C})(X) :\overset{\text{def}}{=} \mathrm{End}(\mathbf{1})$, where $\mathbf{1}$ is the monoidal unit of $\mathsf{Ab}(\mathcal{C}_{X/X})^{\otimes}$. Then

> **Theorem.** The inclusion Alg-type $\hookrightarrow \mathsf{Cat}$ admits a factorization
>
> $$\mathsf{Alg}\text{-type} \xrightarrow{\Psi} \mathsf{Cat}_{/\mathsf{Ring}} \xrightarrow{\text{forgetful}} \mathsf{Cat}$$
>
> such that for any ring R, $\Psi(R) \cong [\mathsf{Alg}_{R/} \xrightarrow{\text{forgetful}} \mathsf{Ring}]$.

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A Problem

Next, we consider the folowing problem:

> Problem. For any reduced rings R, S and $\blacklozenge \subset \{\mathrm{red}\}$, is the following an equivalence?
>
> $$\mathrm{Isom}(R, S) \xrightarrow{\sim} \mathbf{Isom}(\mathsf{Alg}_{R/\blacklozenge}, \mathsf{Alg}_{S/\blacklozenge}),$$
>
> $$f \mapsto f^* :\overset{\text{def}}{=} S \otimes_{f,R} (-).$$

A Problem

First, we note a difference between the present situation and this situation:

Remark. Assume that $\mathrm{red} \in \blacklozenge$.
Then, $\forall A$: R-alg, $\forall M$: A-mod, $A \ltimes M \notin \mathsf{Alg}_{R/\blacklozenge}$.
Hence $\forall A \in \mathsf{Alg}_{R/\blacklozenge}$, $\mathsf{Ab}((\mathsf{Alg}_{R/\blacklozenge})_{A/A}) = 0$.

From another perspective, the above conclude that the following corollary:

Corollary. $R, S \neq 0$: reduced, $\blacklozenge, \lozenge \subset \{\mathrm{red}\}$. Then,
$\mathsf{Alg}_{R/\blacklozenge} \cong \mathsf{Alg}_{S/\lozenge} \Rightarrow \blacklozenge = \lozenge$.

A Problem

Next, we note a property of the tensor product (=push-out) in $\mathsf{Alg}_{R/\blacklozenge}$.

Remark. Assume that $\mathrm{red} \in \blacklozenge$.
Then a push-out of the diagram $[A_1 \leftarrow B \rightarrow A_2]$ in $\mathsf{Alg}_{R/\blacklozenge}$ is naturally isomorphic to $(A_1 \otimes_B A_2)/\sqrt{0}$.
Hence, if $\mathrm{red} \in \blacklozenge$, then $A_1 \otimes_B^{\blacklozenge} A_2 \cong (A_1 \otimes_B A_2)/\sqrt{0}$.

In particular, the operation "push-out" in $\mathsf{Alg}_{R/\blacklozenge}$ preserves surjectivity.
Thus, we obtain the following corollary:

Corollary. Let $[f : A \to B] \in \mathsf{Alg}_{R/\blacklozenge}$ be a morphism. Then
f: surj. $\iff$ the following is a push-out square in $\mathsf{Alg}_{R/\blacklozenge}$:

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\mathrm{pr}_1} & A \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B. \end{array}$$

A Problem

Next, we reduce this problem to the case of the reconstruction problem of schemes.

Definition. X: an R-scheme, $\mathcal{F} : \mathsf{Alg}_{R/\blacklozenge} \to \mathsf{Set}$ a functor.

- $h_X \stackrel{\text{def}}{:=} \mathrm{Hom}_{\mathsf{Sch}_{/R}}(\mathrm{Spec}(-), X) : \mathsf{Alg}_{R/\blacklozenge} \to \mathsf{Set}$.
- For any ring A, $h_A \stackrel{\text{def}}{:=} h_{\mathrm{Spec}(A)}$.
- We shall say that $\mathcal{F}$ is **represented by** X if $F \cong h_X$.

By the above characterization of surjective morphisms, we obtain:

Corollary. Let $A \in \mathsf{Alg}_{R/\blacklozenge}$ be an object, $\mathcal{F} : \mathsf{Alg}_{R/\blacklozenge} \to \mathsf{Set}$ a functor, and $i : \mathcal{F} \to h_A$ a morphism of functors. Then
$\mathcal{F} \to h_A$ is represented by an open subscheme of $\mathrm{Spec}(A) \iff$
$\exists$ a surj:$[A \to B] \in \mathsf{Alg}_{R/\blacklozenge}$ s.t. $\mathcal{F} \times_{h_A} h_B = \varnothing$, and
$\forall \mathcal{G} : \mathsf{Alg}_{R/\blacklozenge} \to \mathsf{Set}$, $\forall (\mathcal{G} \to h_A)$, $\mathcal{G} \times_{h_A} h_B = \varnothing \Rightarrow \exists! (\mathcal{G} \to \mathcal{F})/h_A$.

A Problem

Hence, the property that $\mathcal{F}$ is a sheaf on $\mathsf{Alg}_{R/\blacklozenge}$ with the Zariski topology can be characterized category-theoretically.

Corollary. R: a ring, $\mathcal{F}: \mathsf{Alg}_{R/\blacklozenge} \to \mathsf{Set}$ a Zariski sheaf. Then $\mathcal{F}$ is represented ($\overset{\text{def}}{=}$: repr'd) by an R-scheme that satisfies $\blacklozenge \iff$ $\exists\{A_i\} \subset \mathsf{Alg}_{R/\blacklozenge}$, $\exists\{h_{A_i} \to \mathcal{F}\}_{i\in I}$ s.t., $\forall A \in \mathsf{Alg}_{R/\blacklozenge}$, $\forall(h_A \to \mathcal{F})$,

(1) $\forall i \in I$, $h_{A_i} \times_{\mathcal{F}} h_A \to h_A$ is repr'd by an open subsch of $\mathrm{Spec}(A)$.

(2) $\exists i \in I$, $h_{A_i} \times_{\mathcal{F}} h_A \neq \varnothing$ ($\{U_i\}$ is a "open covering of $\mathcal{F}$").

$\because$) $\mathcal{F}$ is repr'd by a sch $\iff$ $\mathcal{F}$ admits an open cov by repr'ble shvs. $\square$

Thus, the category $\mathsf{Sch}_{\blacklozenge/\,\mathrm{Spec}(R)}$ can be reconstructed from the intrinsic structure of the abstract category $\mathsf{Alg}_{R/\blacklozenge}$.

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Main Theorem

The inclusion $\mathsf{Sch}_{\mathrm{rqqs}}$-type $\hookrightarrow$ Cat admits a factorization

$$\mathsf{Sch}_{\mathrm{rqqs}}\text{-type} \xrightarrow{\Phi} \mathsf{Cat}_{/\,\mathsf{Sch}} \xrightarrow{\text{forgetful}} \mathsf{Cat}$$

such that for any locally Noetherian normal scheme S and set of properties $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$, $\Phi(\mathsf{Sch}_{\blacklozenge/S})$ is isomorphic to the forgetful functor: $\mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}$.

To prove the above theorem, we need to:

Outline. For each object $X \in \mathsf{Sch}_{\blacklozenge/S}$, we give a category-theoretic algorithm for reconstructing the underlying set $|X|$, the topology on $|X|$, and the structure sheaf $\mathcal{O}_X$.

Fiber Product

First, we verify a property of the fiber products in $\mathsf{Sch}_{\blacklozenge/S}$.

> Lemma. $f : Y \to X, g : Z \to X$: morphisms in $\mathsf{Sch}_{\blacklozenge/S}$.
> Suppose that either f or g is quasi-compact.
> Then, the fiber product $Y \times_X^{\blacklozenge} Z$ in $\mathsf{Sch}_{\blacklozenge/S}$ exists, and:
>
> - If $\mathrm{red} \notin \blacklozenge$, then $Y \times_X^{\blacklozenge} Z \cong Y \times_X Z$.
> - If $\mathrm{red} \in \blacklozenge$, then $Y \times_X^{\blacklozenge} Z \cong (Y \times_X Z)_{\mathrm{red}}$.

(We omit a proof).

In particular, $Y \times_X Z$ and $Y \times_X^{\blacklozenge} Z$ have same underlying topological spaces.

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One Pointed Schemes

Next, we note the following:

Observation. A point $x \in X$ is determined by a morphism $f : Y \to X$ s.t. $\#(|Y|) = 1$, and $\mathrm{Im}(f) = \{x\}$.

Hence, the following are equivalent:

- giving a point of X.
- giving a certein equiv class of $f : Y \to X$ s.t. $\#(|Y|) = 1$.

Thus, to reconstruct the underlying sets of the objects of $\mathsf{Sch}_{\blacklozenge/S}$, it suffices to characterize one-pointed schemes (i.e., schemes whose underlying sets are one point sets) category-theoretically.

One Pointed Schemes

By considering the above observation, we can give a catgrory-theoretic characterization of one-pointed objects as follows:

Lemma. Let $X \in \mathsf{Sch}_{\blacklozenge/S}$. Then $\#(|X|) > 1 \iff \exists Y, Z \neq \varnothing$, $\exists Y \to X, Z \to X$ s.t. $Y \times_X^{\blacklozenge} Z = \varnothing$

$\because$) [$\Rightarrow$] $\exists x_1, x_2 \in X$ s.t. $x_1 \neq x_2 \Rightarrow \mathrm{Spec}(k(x_1)) \times_X^{\blacklozenge} \mathrm{Spec}(k(x_2)) = \varnothing$.
[$\Leftarrow$] If X satisfies the condition, then $y \in Y$ and $z \in Z$ determine two distinct points of X. $\square$

Now, we can obtain immediately the following:

Corollary. Let $\mathcal{C} \in \mathsf{Sch}_{\mathrm{rqqs}}$-type be a category of scheme type and $X \in \mathcal{C}$ an object.
Assume that $\exists F : \mathcal{C} \xrightarrow{\sim} \mathsf{Sch}_{\blacklozenge/S}$ s.t. $\#(|F(X)|) = 1$.
Then $\forall F : \mathcal{C} \to \mathsf{Sch}_{\blacklozenge/S}$, $\#(|F(X)|) = 1$.

Underlying Sets

Let $\mathcal{C} \in \mathsf{Sch}_{\mathrm{rqqs}}$ -type be a category of schematic type and $X \in \mathcal{C}$.
We call that X is **1-ptd** if $\exists F : \mathcal{C} \xrightarrow{\sim} \mathsf{Sch}_{\blacklozenge/S}, \#(|F(X)|) = 1$.

Remark. By the above corollary, this definition does not depend on the choice of F.

Definition. Let $\mathcal{C}$ be a categiry of schematic type and $X \in \mathcal{C}$. We define

$$\mathrm{Pt}_{\mathcal{C}}(X) :\overset{\text{def}}{=} \{(p_Z : Z \to X) \in \mathcal{C} \mid Z\text{: 1ptd}\} / \sim,$$

where $(p_Z : Z \to X) \sim (p_{Z'} : Z' \to X) \ :\overset{\text{def}}{\Longleftrightarrow} \ Z \times^{\mathcal{C}}_{p_Z, X, p_{Z'}} Z' \neq \varnothing$. By compositing a morphism $f : X \to Y$ in $\mathcal{C}$, we obtain a map $\mathrm{Pt}_{\mathcal{C}}(f) : \mathrm{Pt}_{\mathcal{C}}(X) \to \mathrm{Pt}_{\mathcal{C}}(Y)$.

Thus, we obtain a functor $\mathrm{Pt}_{\mathcal{C}} : \mathcal{C} \to \mathsf{Set}$. Moreover,

Underlying Sets

Reconstruction of Underlying Sets.
The assignment $\Psi_{\mathsf{Set}} : \mathcal{C} \mapsto \mathrm{Pt}_{\mathcal{C}}$ determines a factorization

$$\mathsf{Sch}_{\mathrm{rqqs}}\text{-type} \xrightarrow{\Psi_{\mathsf{Set}}} \mathsf{Cat}_{/\,\mathsf{Set}} \xrightarrow{\text{forgetful}} \mathsf{Cat}$$

of the inclusion $\mathsf{Sch}_{\mathrm{rqqs}}$-type $\hookrightarrow$ Cat such that
for any scheme S and $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$,
$\Psi_{\mathsf{Set}}(\mathsf{Sch}_{\blacklozenge/S}) : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Set}$ is isomorphic to $\mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\text{forgetful}} \mathsf{Set}$.

In particular,

Corollary. For any $F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$, $U^{\mathsf{Set}}_{\blacklozenge/S} \cong U^{\mathsf{Set}}_{\lozenge/T} \circ F$.

$$\begin{array}{ccc} \mathsf{Sch}_{\blacklozenge/S} & \xrightarrow[\sim]{F} & \mathsf{Sch}_{\lozenge/T} \\ {\scriptstyle U^{\mathsf{Set}}_{\blacklozenge/S}}\downarrow & & \downarrow{\scriptstyle U^{\mathsf{Set}}_{\lozenge/T}} \\ \mathsf{Set} & = & \mathsf{Set}\,. \end{array}$$

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Closed Immersions

To reconstruct the underlying topological spaces, we use a property of regular monomorphisms in $\mathsf{Sch}_{\blacklozenge/S}$.

Definition. Let $\mathcal{C}$ be a category and $(f : X \to Y) \in \mathcal{C}$.
We call f is a **regular monomorphism** if $\exists g, h : Y \to Z$, s.t. f is the equalizer of (g, h).

Lemma. Let S be a qsep scheme and $f \in \mathsf{Sch}_{\blacklozenge/S}$ a morphism.
If f is a regular monomorphism in $\mathsf{Sch}_{\blacklozenge/S}$, then f: immersion.

∵) If f is a reg. mono., then f is isomorphic to the base-change of the diagonal of some $g \in \mathsf{Sch}_{\blacklozenge/S}$ (details omitted). □

A Characterization of Reduced Schemes. Let S: qsep.
$X \in \mathsf{Sch}_{\blacklozenge/S}$ is red. $\iff$ [$f : Y \to X$: surj. reg. mono. $\Rightarrow$ f: isom.]

∵) a surj. reg. mono. is a surj. closed immersion. □

Closed Immersions

Closed immersions may be characterized as follows:

> A Characterization of Closed Immersions. [vDdB19]
> S: q.s., $(f : X \to Y) \in \mathsf{Sch}_{\blacklozenge/S}$. Then f is a closed immersion $\iff$
>
> - f is a regular monomorphism.
> - $\forall(T \to Y)$, the base-change $X_{\blacklozenge,T} = X \times_Y^{\blacklozenge} T$ exists in $\mathsf{Sch}_{\blacklozenge/S}$.
> - $\forall(T \to Y)$, $\forall t \in T$: closed pts. s.t. $t \notin \mathrm{Im}(f_{\blacklozenge,T} : X_{\blacklozenge,T} \to T)$, $X_{\blacklozenge,T} \coprod \mathrm{Spec}(k(t)) \to T$ is a regular monomorphism.

(We omit a proof).

Hence to give a category-theoretic characterization of closed imms., it suffices to characterize the closed points of each objects.
In particular, it suffices to characterize the specializaton-generization relation $x_1 \rightsquigarrow x_2$.

Strongly Local

To characterize the relation $x_1 \rightsquigarrow x_2$, we define:

Definition. S: qsep, $X \in \mathsf{Sch}_{\blacklozenge/S}$, and $x_1, x_2 \in X$.
(X, x_1, x_2) is **strongly local** in $\mathsf{Sch}_{\blacklozenge/S}$ $:\overset{\text{def}}{\Longleftrightarrow}$

- X is connected.
- $\forall(f : Z \to X)$: reg. mono., if $x_1, x_2 \in \mathrm{Im}(f)$, then f: isom.
- $\mathrm{Spec}(k(x_1)) \coprod \mathrm{Spec}(k(x_2)) \to X$ is an epimorphism.
- $\mathrm{Spec}(k(x_1)) \to X$ is a regular monomorphism.
- $\forall(f : Z \to X)$: reg. mono., if $x_1 \notin \mathrm{Im}(f)$ and $Z \neq \varnothing$, then $Z \coprod \mathrm{Spec}(k(x_1)) \to X$ is **not** a regular monomorphism.

Remark. The property that (X, x_1, x_2) is strongly local is defined category-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x_1, x_2)$.

Strongly Local

Properties of strongly local objects which is used to characterize the relation $\rightsquigarrow$ are followings:

Properties of Strongly Local Objects.
Let S be a qsep scheme, $X \in \mathsf{Sch}_{\blacklozenge/S}$, and $x_1, x_2 \in X$.
If (X, x_1, x_2): strongly local in $\mathsf{Sch}_{\blacklozenge/S}$, then

(1) $X \cong \mathrm{Spec}$(a local domain)

(2) One of x_1, x_2 is the closed pt., and the other is the generic pt.

In particular, $x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$.

(We omit a proof)

Let $V :\overset{\text{def}}{=} \mathrm{Spec}$(a val. ring), $v \in V$: closed pt., $\eta \in V$: generic pt. Then,

Example. (V, v, η): strongly local.

(We omit a proof)

Relation "$x_1 \rightsquigarrow x_2$"

S: q.s., $X \in \mathsf{Sch}_{\blacklozenge/S}$, $x_1, x_2 \in X$.

A Characterization of "$x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$".

"$x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$" $\iff$
$\exists Z \in \mathsf{Sch}_{\blacklozenge/S}, \exists z_1, z_2 \in Z, \exists (f : Z \to X) \in \mathsf{Sch}_{\blacklozenge/S}$, s.t.,
(Z, z_1, z_2): str. loc., and $\{f(z_1), f(z_2)\} = \{x_1, x_2\}$.

$\because$) [$\Rightarrow$] Take a valuation ring which dominates $\mathcal{O}_{X,x_1}/\mathfrak{m}_{X,x_2}$.
[$\Leftarrow$] follows from the previous properties. $\square$

By using the above characterization, we can characterize category-theoretically of the relation $x_1 \rightsquigarrow x_2$ (details omitted).

Corollary. We conclude the followings:

- Closed immersions may be characterized cat.-theoretically.
- Underlying top. may be reconstructed cat.-theoretically.

Underlying Top.

Similarly to the case of Set, we conclude:

Reconstruction of Underlying Tops.

There exists a factorization

$$\mathsf{Sch}_{\mathrm{rqqs}}\text{-type} \xrightarrow{\Psi_{\mathsf{Top}}} \mathsf{Cat}_{/\,\mathsf{Top}} \xrightarrow{\mathrm{forget}} \mathsf{Cat}$$

of the inclusion $\mathsf{Sch}_{\mathrm{rqqs}}$-type $\hookrightarrow \mathsf{Cat}$ such that

for any scheme S and $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$,

$\Psi_{\mathsf{Top}}(\mathsf{Sch}_{\blacklozenge/S}) : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Top}$ is isomorphic to $\mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\mathrm{forget}} \mathsf{Top}$.

Thus, in particular, $\forall F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$, $U^{\mathsf{Top}}_{\blacklozenge/S} \cong U^{\mathsf{Top}}_{\lozenge/T} \circ F$.

Corollary. Topological properties of schemes (or morphisms) may be characterized cat.-theoretically (ex: q.s., q.c., sep., irred., local ($\cong \mathrm{Spec}$(local ring)), open imm., univ. closed, etc.).

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An Observation

To reconstruct the structure sheaf of $X \in \mathsf{Sch}_{\blacklozenge/S}$, it suffices to characterize the ring scheme $\mathbb{A}^1_X \to X$ category-theoretically.
Since $\mathbb{A}^1$ is of finite presentation over the base scheme, we wont to give a cat.-theoretic characterization of morphisms of finite presentation

An Idea. of f.p.$/S \risingdotseq$ a ``compact object" in $\mathsf{Sch}^{\mathrm{op}}_{/S}$

More precisely,

A Characterization. $X \to S$ is of finite presentation $\iff$
$\forall (V_\lambda, f_{\lambda\mu})_{\lambda \in \Lambda}$: diagram in $\mathsf{Sch}_{/S}$ s.t. Λ: cofiltered, V_λ: affine,
the following natural map is surjective:

$$\varphi : \operatorname*{colim}_{\lambda \in \Lambda^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{/S}}(V_\lambda, X) \twoheadrightarrow \mathrm{Hom}_{\mathsf{Sch}_{/S}}(\lim_{\lambda \in \Lambda}{}^{\blacklozenge} V_\lambda, X).$$

Note that we don't have a characterization of affine schemes yet.

Essentially of F.P.

By consulting to the previous characterization, we will give a characterization of morphisms that is locally of finite presentation.

S: locally Noetherian, $(f : X \to S) \in \mathsf{Sch}_{\blacklozenge/S}$, $x \in X$.

Lemma. $f_x^\# : \mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$: essentially of finite presentation
$\iff \forall (V_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$: diagram in $\mathsf{Sch}_{\blacklozenge/S}$ s.t.
Λ: cofiltered, V_λ: local, $f_{\lambda\mu}(\text{closed pt.}) = f(x)$,
the following natural map is surjective :

$$\operatorname*{colim}_{\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(V_\lambda, X) \twoheadrightarrow \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/S}}(\lim_{\lambda\in\Lambda}{}^{\blacklozenge} V_\lambda, X).$$

$\because$) It can be proved by a same method of the proof of the previous characterization (details omitted). $\square$

Remark. I think that if S is not stalkwise Noetherian, then there exists a counterexample of the above Lemma.

Locally of F.P.

S: locally Noetherian, $(f : X \to S) \in \mathsf{Sch}_{\blacklozenge/S}$.

Lemma. f is locally of finite presentation $\iff$

- $\forall x \in X$, $f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$: essentially of f.p.
- $\forall (Z \to Y)$, $\forall z \in Z$, the following natural map is bijective :

$$\operatorname*{colim}_{W \in I_Z(z)^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(W, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(\lim_{W \in I_Z(z)}{}^{\blacklozenge} W, X),$$

where $I_Z(z) :\overset{\text{def}}{=} \{i_W : W \to Z \mid i_W\text{: open imm., } z \in \mathrm{Im}(i_W)\}$.

∵) It can be proved by a same method of the proof of the previous characterization (details omitted). □

List

Let S be a locally Noetherian scheme. For any $X \in \mathsf{Sch}_{\blacklozenge/S}$, $|X|$ has been reconstructed cat.-theoretically, and, moreover:

List of Category-Theoretic Properties.

The following properties have been characterized cat.-theoretically:

- red., irred., integral, q.c., $\cong \mathrm{Spec}$(local ring), $\cong \mathrm{Spec}$(field).
- q.c., q.s., sep., imm., closed imm., open imm., loc. of f.p., f.p., f.p. + proper ($=$ sep.$+$ f.p.$+$ univ. closed).

The following properties have not given yet cat.-theoretic characterizations:

flat, smooth, étale, unramified, etc.

An Observation

To reconstruct the structure sheaf of $X \in \mathsf{Sch}_{\blacklozenge/S}$,
it suffices to characterize the ring scheme $\mathbb{A}^1_S \to S$ cat.-theoretically.

Since $\mathbb{A}^1_S = \mathbb{P}^1_S \setminus \{\infty\}$,
it suffices to characterize $\mathbb{P}^1_S \to S$ category-theoretically.

What is Needed.

Giving a category-theoretic characterizaion of $\mathbb{P}^1_S \in \mathsf{Sch}_{\blacklozenge/S}$

To do this, we give a category-theoretic characterization of $\mathbb{P}^1_k$.

Projective Line

First, we note that:

$$\mathbb{P}^1_k \iff \begin{cases} \text{- proper over } \mathrm{Spec}(k) \\ \text{- the residue field of the generic pt.} \cong k(t) \\ \text{- "Closest" to } \mathrm{Spec}(k(t)) \end{cases}$$

Hence, to characterize $\mathbb{P}^1_k$, it suffices to characterize $\mathrm{Spec}(k(t))$.

Idea: Lüroth's theorem.

> Lemma. $[f : Y \to \mathrm{Spec}(k)] \cong [\mathrm{Spec}(k(t)) \to \mathrm{Spec}(k)] \iff$
>
> - $\exists K$: field , $Y \cong \mathrm{Spec}(K)$
> - f: not f.p. ($\Leftrightarrow K/k$: not a finite extension)
> - $k \subsetneq \forall L \subset K, \exists$ isom. $K \cong L$ over k (Lüroth's theorem).

Thus, we obtain a category-theoretic characterization of $\mathbb{P}^1_k$.

An Observation

$\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ has a ring scheme structure:

Observation. 1-dim ring scheme $\risingdotseq \mathbb{A}^1$.

Indeed, we can prove the following:

Lemma ♠. $V :\stackrel{\text{def}}{=} \operatorname{Spec}(\text{DVR})$, $f : X \to V$: a flat ring scheme/V.
Then f is isomorphic to the projection $\mathbb{A}^1_V \to V$ if

- The special fiber of f is connected and 1-dim.
- The generic fiber of f is $\mathbb{A}^1_\eta$, where $\eta \in V$ is the generic point.

$\because$) This can be proved by a same method of the proof of the well-known fact of the theory of Néron models that $\mathbb{G}_{m,V}$ is an absolute minimal model. □

Remark. Without connectedness of the special fiber, there is a counterexample: $\operatorname{Spec}(R[x, (x^{p^2} - x^p)/\pi])$.

Characterization of $\mathbb{P}^1_S$

S: locally Noetherian **normal**, $\Diamond = \blacklozenge \cup \{\mathrm{red}\}$,
$f : X \to S$ a morphism in $\mathsf{Sch}_{\blacklozenge/S}$. Then
f is isomorphic to the projection $\mathbb{P}^1_S \to S \iff f$ satisfies:

(1) f is proper.

(2) $\forall s \in S$, $X \times_S^{\blacklozenge} k(s)\ (= f^{-1}(s)_{\mathrm{red}}) \cong \mathbb{P}^1_{k(s)}$.

(3) $\forall$generic pt. $\eta \in S$, $X \times_S^{\blacklozenge} k(\eta)\ (= f^{-1}(\eta)) \cong \mathbb{P}^1_{k(\eta)}$.

(4) $\exists s_0, s_1, s_\infty$: sections of f s.t. $s_i \cap s_j = \varnothing, (i \neq j)$.

(5) $\forall i = 0, 1, \infty$, $\exists$ a ring structure on $X \setminus s_i$ over S in $\mathsf{Sch}_{\Diamond/S}$ s.t. s_j: add. unit, s_k: mult. unit, and $\{i, j, k\} = \{0, 1, \infty\}$.

(6) $\forall (g : Y \to S) \in \mathsf{Sch}_{\blacklozenge/S}$ and $\forall (t_0, t_1, t_\infty)$: sections of g, if $(g; t_0, t_1, t_\infty)$ satisfy (1),...,(5), then $\exists! h : X \to Y$: closed imm. s.t. $f = g \circ h, h \circ s_i = t_i, \forall i \in \{0, 1, \infty\}$. (universality)

Proof

If $\mathbb{P}^1_S$ satisfies (6), then by the uniqueness of (6), "$\Leftarrow$": ok.
Hence, it suffices to prove "$\Rightarrow$" (i.e., $\mathbb{P}^1_S$ satisfies (6)).

Let $Y \in \mathsf{Sch}_{\blacklozenge/S}$ be an object that satisfies (1),...,(5). We define

$$C : \mathsf{Sch}_{/S}^{\mathrm{op}} \to \mathsf{Set},$$
$$(T \to S) \mapsto \left\{ i : \mathbb{P}^1_T \to Y_T \;\middle|\; \begin{array}{c} i\text{: closed imm., s.t.,} \\ 0, 1, \infty \mapsto t_0, t_1, t_\infty \end{array} \right\}.$$

Then we obtain immediately that:

- C is an algebraic space over S locally of finite type
- by (2), each fiber of $C \to S$ is a 1-pt. set.
- by (3), $C \to S$ is birational.

Assertion. The cardinality of $C(S)$ is one.

Proof

To prove that $\#(C(S)) = 1$, it suffices to prove that

the composite $C_{\mathrm{red}} \hookrightarrow C \to S$ is an isomorphism.

Let $W \overset{\text{def}}{:=} \mathrm{Spec}(\text{DVR}) \to S$ be an morphism (note that $W \in \mathsf{Sch}_{\blacklozenge/S}$).
Then $(Y_W)_{\mathrm{red}} \setminus t_{i,W}$ is a flat ring scheme $/W$.
By **Lemma ♠**, we conclude that

$(Y_W)_{\mathrm{red}} \setminus t_{i,W} \cong \mathbb{A}^1_W$. In particular, $\#(C(W)) = 1$.

By the valuative criterion, $C \to S$ is proper. Since $C \to S$ is bijective,

$C \to S$ is finite. In particular, C is a **scheme**.

Since $C \to S$ is birational, and S is normal, $C_{\mathrm{red}} \xrightarrow{\sim} S$.
This completes the proof of the "**Characterization of $\mathbb{P}^1_S$**". $\square$

Conclusion

Similarly to the case of Set and Top, we conclude:

Reconstruction of Underlying Schemes.

There exists a factorization

$$\mathsf{Sch}_{\mathrm{rqqs}}\text{-type} \xrightarrow{\Psi_{\mathsf{Sch}}} \mathsf{Cat}_{/\,\mathsf{Sch}} \xrightarrow{\text{forgetful}} \mathsf{Cat}$$

of the inclusion $\mathsf{Sch}_{\mathrm{rqqs}}$-type $\hookrightarrow$ Cat such that for any locally Noetherian normal S and $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$, $\Psi_{\mathsf{Sch}}(\mathsf{Sch}_{\blacklozenge/S}) : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}$ is isom to $\mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\text{forgetful}} \mathsf{Sch}$.

Thus, in particular, $\forall F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$, $U^{\mathsf{Sch}}_{\blacklozenge/S} \cong U^{\mathsf{Sch}}_{\lozenge/T} \circ F$.

Moreover, for any locally Noetherian normal schemes S, T,

$$\mathrm{Isom}(S, T) \xrightarrow{\sim} \mathbf{Isom}(\mathsf{Sch}_{\blacklozenge/T}, \mathsf{Sch}_{\blacklozenge/S}).$$

Related Works

I also confirmed the following reconstruction results:

- $\blacklozenge \subset \{\mathrm{finite},\ \mathrm{proper}\}$ and S: Noetherian
- $\blacklozenge \subset \{\mathrm{ft},\ \mathrm{red},\ \mathrm{qcpt,qsep,sep}\}$ and S: locally Noetherian normal, where "ft" means "of finite type".
- The log-scheme version of the above reconstruction problem.

Since we may consider many properties of schemes,
there are many category-theoretic reconstruction problems.

THANK YOU

FOR YOUR ATTENTION

$$
\begin{array}{ccc}
\mathsf{Sch}_{\mathrm{rqqs}}\text{-type} & \xrightarrow{\ {}^{\exists}\Psi_{\mathsf{Sch}}\ } & \mathsf{Cat}_{/\,\mathsf{Sch}} \\
\Big\downarrow \text{inclusion} & \circlearrowright & \Big\downarrow \text{forgetful} \\
\mathsf{Cat} & = & \mathsf{Cat}
\end{array}
$$

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