EXTRIANGULATED CATEGORIES, LOCALIZATIONS, AND RELATIVE THEORIES

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Convention

We assume the followings:

- All subcategories are full and closed under isomorphisms.
- $\mathcal{C}$: a skeletally small additive category.
  $\rightsquigarrow \operatorname{Mod}\mathcal{C}$: the category of all right $\mathcal{C}$-modules, that is,
  contravariant additive functors $\mathcal{C}^{\mathrm{op}} \to Ab$.

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Extriangulated categories introduced by Nakaoka-Palu unify exact categories, triangulated categories and their extension-closed subcategories, and enable us to deal them via homological algebra.

Aim of this section

- Introduce extriangulated categories.
- See some examples.

Let us start to recall exact and triangulated categories.

Definition

An exact category consists of

- $\mathcal{C}$: an additive category
- $\mathcal{E}$: conflations, a class of kernel-cokernel pairs

We consider a biadditive functor $\mathrm{Ext}^1_{\mathcal{E}}(-,-)$. For any $\delta \in \mathrm{Ext}^1_{\mathcal{E}}(C, A)$, we obtain a conflation

$$\mathfrak{s}(\delta)\colon\ A \rightarrowtail B \twoheadrightarrow C.$$

Definition

A triangulated category consists of

- $\mathcal{C}$: an additive category
- $[1]$: an autoequivalence functor on $\mathcal{C}$
- $\mathcal{E}$: a class of distinguished triangles $X \to Y \to Z \to X[1]$

We consider a biadditive functor $\mathcal{C}(-, -[1])$. For any $\delta \in \mathcal{C}(C, A[1])$, we obtain a distinguished triangle.

$$\mathfrak{s}(\delta) \colon A \to B \to C \xrightarrow{\delta} A[1].$$

In both cases, there exist the biadditive functor

$$\mathbb{E} = \mathrm{Ext}^1_{\mathcal{C}}(-,-) \text{ or } \mathcal{C}(-,-[1]) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$$

and the map $\mathfrak{s}$

$$\mathbb{E}(C,A) \ni \delta \overset{\mathfrak{s}}{\rightsquigarrow} A \to B \to C (\xrightarrow{\delta} A[1])$$

We simultaneously generalize the above observation.

$\mathcal{C}$: an additive category, $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$: a biadditive functor
For any $\delta \in \mathbb{E}(C, A)$, we associate to an equivalence class of sequences of morphisms $[A \to B \to C]$, denoted by $\mathfrak{s}(\delta)$. Here the equivalence relation is described as the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "{}^{\exists}b"] & C \arrow[d, equal] \\ A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \end{tikzcd}$$

b: an isomorphism.

We prepare some notations.
$a\colon A \to A', c\colon C' \to C$ in $\mathcal{C}$

- $a_* := \mathbb{E}(-, a)\colon \mathbb{E}(-, A) \to \mathbb{E}(-, A')$
- $c^* := \mathbb{E}(c, -)\colon \mathbb{E}(C, -) \to \mathbb{E}(C', -)$

and

- We call $\delta \in \mathbb{E}(C, A)$ an $\mathbb{E}$-extension and denote it by $C \overset{\delta}{\dashrightarrow} A$.
- A pair of $\delta \in \mathbb{E}(C, A)$ and a representative element $A \to B \to C$ of $\mathfrak{s}(\delta) = [A \to B \to C]$ is called an $\mathfrak{s}$-triangle, and denoted by $A \to B \to C \overset{\delta}{\dashrightarrow}$.

$\mathfrak{s}$: an additive realization $:\Leftrightarrow$ It satisfies

- Suppose $a_*\delta = c^*\delta'$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, dashed, "\delta'"'] & A'
\end{tikzcd}$$

- $\mathfrak{s}(0)$: a split exact sequence for $0 \in \mathbb{E}(C, A)$.

$\mathfrak{s}$: an additive realization $:\Leftrightarrow$ It satisfies

- Suppose $a_*\delta = c^*\delta'$. Then there exists b as the following.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, dashed, "\delta'"'] & A' \end{tikzcd}$$

- $\mathfrak{s}(0)$: a split exact sequence for $0 \in \mathbb{E}(C, A)$.

- Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any elements and suppose $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{f'} B' \xrightarrow{g'} C']$.
  Define $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ as the element correoponding to $(\delta, 0, 0, \delta') \in \mathbb{E}(C, A) \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A')$ via the natural isomorphism.
  Then $\mathfrak{s}(\delta \oplus \delta') = [A \oplus A' \xrightarrow{f \oplus f'} B \oplus B' \xrightarrow{g \oplus g'} C \oplus C']$ holds.

An extriangulated category is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfying (ET1), (ET2), (ET3), (ET3)$^{\mathrm{op}}$, (ET4), (ET4)$^{\mathrm{op}}$.

(ET1) $\mathbb{E}$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization.

(ET3) Suppose $f'a = bf$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] & {} \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, dashed, "\delta'"'] & {} \end{tikzcd}$$

An extriangulated category is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfying (ET1), (ET2), (ET3), $(\mathrm{ET3})^{\mathrm{op}}$, (ET4), $(\mathrm{ET4})^{\mathrm{op}}$.

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A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, dashed, "\delta'"'] & {}
\end{tikzcd}$$

and $c^*\delta' = a_*\delta$ holds.

(ET4)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"'] & C \arrow[r, dashed, "\delta"] & {} \\
A \arrow[r, "gf"'] & C \arrow[d, "g'"'] & & \\
& F \arrow[d, dashed, "\gamma"] & & \\
& {} & &
\end{tikzcd}$$

(ET4) Then there exists the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "d"] & {} \\
A \arrow[r, "gf"'] & C \arrow[r, "h'"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, dashed, "\delta'"'] \arrow[d, "e"] & {} \\
& F \arrow[r, equal] \arrow[d, dashed, "\gamma"] & F \arrow[r, dashed, "\gamma"'] \arrow[d, dashed, "f'\gamma"] & {} \\
& {} & {} &
\end{tikzcd}$$

and $d^*\delta' = \delta$ and $e^*\gamma = f_*\delta'$ hold.

$f\colon A \to B$ (resp. $g\colon B \to C$) is an $\mathfrak{s}$-inflation (resp. $\mathfrak{s}$-deflation)
$:\Leftrightarrow {}^{\exists}A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$: an $\mathfrak{s}$-triangle
We denote by $\rightarrowtail$ and $\twoheadrightarrow$ an $\mathfrak{s}$-inflation and an $\mathfrak{s}$-deflation, respectively.

Example

- Exact categories are extriangulated categories.
- Conversely, any extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ in which any $\mathfrak{s}$-inflation is mono and any $\mathfrak{s}$-deflation is epi is an exact category. More precisely, the class of all $\mathfrak{s}$-triangles satisfies the axiom of exact categories.

Example

- Triangulated categories are extriangulated categories.
- Conversely, any extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ in which any morphism is both an $\mathfrak{s}$-inflation and an $\mathfrak{s}$-deflation is a triangulated category. More precisely, a class of all $\mathfrak{s}$-triangles satisfies the axiom of triangulated categories.

Proposition (Nakaoka-Palu)

For any $\mathfrak{s}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$, we obtain an exact sequence

$$\begin{aligned} \mathcal{C}(-,A) \xrightarrow{f\circ -} \mathcal{C}(-,B) \xrightarrow{g\circ -} \ & \mathcal{C}(-,C) \\ \xrightarrow{\delta_\sharp} \ & \mathbb{E}(-,A) \xrightarrow{f_*} \mathbb{E}(-,B) \xrightarrow{g_*} \mathbb{E}(-,C) \end{aligned}$$

in $\operatorname{Mod}\mathcal{C}$. *Here $\delta_\sharp$ is the morphism corresponding to $\delta \in \mathbb{E}(C,A)$ by the Yoneda lemma.*

Definition

- $P \in \mathcal{C}$ is projective $:\Leftrightarrow \mathbb{E}(P, \mathcal{C}) = 0$.
- $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has enough projective
  $:\Leftrightarrow {}^{\forall}C \in \mathcal{C}\ {}^{\exists}(A \to P \to C \dashrightarrow)$: $\mathfrak{s}$-triangle, P: projective
- Dually, injective and enough injective are defined.

$$\begin{tikzcd}
 & & P \arrow[dl, dashed, "\exists"'] \arrow[d, "\forall"] & & \\
A \arrow[r] & B \arrow[r] & C \arrow[r, dashed] & {} & : \mathfrak{s}\text{-triangle}
\end{tikzcd}$$

Proposition (Nakaoka-Palu)

$\mathcal{I} \subseteq \mathcal{C}$: a subcategory consisting of projective and injective objects
Then the ideal quotient $\underline{\mathcal{C}}$ by $\mathcal{I}$ has a natural extriangulated structure.

This generalizes the construction of a triangulated category from a Frobenius exact category.

There are the following way to obtain a new extriangulted category.

- ideal quotients (previous)
- extension-closed subcategories (next)
- localizations (Section 2)
- relative theories (Section 3)

Example

$(\mathcal{C}, \mathbb{E}, \mathfrak{s})$: an extriangulated category, $\mathcal{D} \subseteq \mathcal{C}$: an additive subcategory closed under extensions, that is, for any $\mathfrak{s}$-triangle $A \to B \to C \dashrightarrow$,

$$A, C \in \mathcal{D} \Rightarrow B \in \mathcal{D}.$$

Then $(\mathcal{D}, \mathbb{E}|_{\mathcal{D}}, \mathfrak{s}|_{\mathcal{D}})$: an extriangulated category. In particular, extension-closed subcategories of triangulated categories are extariangulated categories.

Recent years, there are several studies in which extriangulated categories naturally appear as extension-closed subcategories of triangulated categories, for example,

- Cohen-Macaulay DG modules [Jin]
- extended cohearts of co t-structures [Pauksztello-Zvonareva]

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The following examples are classical results about localizations and fuondamental tools even now.

Example (Serre quotient)

$\mathcal{A}$: an abelian category, $\mathcal{S} \subseteq \mathcal{A}$: a Serre subcategory
$\rightsquigarrow \mathcal{A}/\mathcal{S}$: an abeian category and $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$: an exact functor.

Example (Verdier quotient)

$\mathcal{T}$: a triangulated category, $\mathcal{U} \subseteq \mathcal{A}$: a thick subcategory
$\rightsquigarrow \mathcal{T}/\mathcal{U}$: a triangulated category and $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{U}$: a triangulated functor.

Aim of this section
$(\mathcal{C}, \mathbb{E}, \mathfrak{s})$: an extriangulated category, $\mathcal{N} \subseteq \mathcal{C}$: a thick subcategory. with some assumptions.
$\rightsquigarrow \mathcal{C}/\mathcal{N}$: an extriangulated category and $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$: an extriangulated functor.

This is a joint work with Hiroyuki Nakaoka (Nagoya University) and Yasuaki Ogawa (Nara University of Education).

Definition (Bennett-Tennenhaus-Shah)

$(\mathcal{C}, \mathbb{E}, \mathfrak{s}), (\mathcal{D}, \mathbb{F}, \mathfrak{t})$: extriang. categories, $F\colon \mathcal{C} \to \mathcal{D}$: an additive functor
$\eta\colon \mathbb{E}(-,-) \to \mathbb{F}(F-, F-)$: a natural transformation.
(F, η): an extriangulated functor $:\Leftrightarrow \mathfrak{t}(\eta_{C,A}(\delta)) = [FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC]$
for any $\mathfrak{s}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$.

$$\begin{tikzcd}
(A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "F"] & C \arrow[r, dashed, "\delta"] &) & \delta \in \mathbb{E}(C,A) \arrow[l, squiggly, "\mathfrak{s}"'] \arrow[d, "\eta_{C,A}"] \arrow[dl, phantom, "\circlearrowright"] \\
(FA \arrow[r, "Ff"] & FB \arrow[r, "Fg"] & FC \arrow[r, dashed, "\eta_{C,A}(\delta)"] &) & \eta_{C,A}(\delta) \in \mathbb{F}(FC, FA) \arrow[l, squiggly, "\mathfrak{t}"']
\end{tikzcd}$$

Example

- exact functors $\fallingdotseq$ extriangulated functors between exact categories
- triangle functors $\fallingdotseq$ extriangulated functors between triangulated categories

Definition

$\mathcal{N}$: an additive subcategory of $\mathcal{C}$

$\mathcal{N}$: a thick subcategory $:\Leftrightarrow$ It satisfies

- $\mathcal{N}$ is closed under taking direct summands.
- The two out of three property with respect to any $\mathfrak{s}$-triangle, that is, for any $\mathfrak{s}$-triangle $A \to B \to C \dashrightarrow$, if two of A, B and C belong to $\mathcal{N}$, then so does the third.

Note that thick subcategories coincide with usual thick subcategories when $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a triangulated category.

Definition

$\mathcal{N}$: a thick subcategory of $\mathcal{C}$

$\mathcal{N}$: a biresolving subcategory $:\Leftrightarrow$ It satisfies

- For any $C \in \mathcal{C}$, there is an $\mathfrak{s}$-deflation $N \twoheadrightarrow C$ with $N \in \mathcal{N}$.
- For any $C \in \mathcal{C}$, there is an $\mathfrak{s}$-inflation $C \rightarrowtail N$ with $N \in \mathcal{N}$.

Note that biresolving subcategories coincide with thick subcategories when $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a triangulated category.

Theorem (Nakaoka-Ogawa-S)

*$\mathcal{N}$: a biresolving subcat. of $\mathcal{C}$, $\mathcal{S}_{\mathcal{N}} := \{\overset{f}{\rightarrowtail}\overset{g}{\twoheadrightarrow} \mid \operatorname{Cone} f, \operatorname{CoCone} g \in \mathcal{N}\}$
Then the localization $\mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ has a natural triangulated structure such that the localization functor $Q \colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ is an extriangulated functor.*

This result recovers the Verdier quotient, a localization of an extriangulated category by a Hovey twin cotorsion pair [Nakaoka-Palu] and a localization of an exact category by a biresolving subcategory [Rump].

Definition

$\mathcal{N}$: a thick subcat. of $\mathcal{C}$
$\mathcal{N}$: a percolating subcategory $:\Leftrightarrow$ It satisfies the following condition and its dual.

- For any $f : N \to C$ with $N \in \mathcal{N}$, there is a factorization

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, two heads] & & C \\ & N' \arrow[ur, tail] & \end{tikzcd}$$

with $N' \in \mathcal{N}$.

- When $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is abelian, percol. subcat. $=$ Serre subcat.
- When $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is triang, percol. subcat. $=$ thick subcat.

Theorem (Nakaoka-Ogawa-S)

$\mathcal{N} \subseteq \mathcal{C}$: a percolating subcategory with some technical assumptions

$\mathcal{S}_{\mathcal{N}} := \{ \overset{f}{\twoheadrightarrow} \overset{g}{\rightarrowtail} \mid \mathrm{CoCone}\, f, \mathrm{Cone}\, g \in \mathcal{N} \}$

Then the localization $\mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ has a natural extriangulated sructure such that the localization functor $Q \colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ is an extriangulated functor.

This covers the Serre quotient, the Verdier quotient and a localization of an exact category by a two-sided admissibly percolating subcategory. [Henrard-Roosmalen].

Remark

In [Nakaoka-Ogawa-S], we give a sufficient condition of a class of morphisms such that the localization becomes an extriangulated category.

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relative theories $=$ substructures of exact (extriangulated) categories

Relative theories have been studied in

- module categories over artin algebras [Auslander-Solberg]
- exact categories [Draxler-Reiten-Smalø-Solberg-Keller]
- extriangulated categories [Herschend-Liu-Nakaoka]

$\mathcal{C}$: an additive category

$$\begin{array}{ccc} \boxed{\text{exact structure on } \mathcal{C}} & \longleftrightarrow & \boxed{\text{Yoneda } \mathrm{Ext} \text{ group } \mathrm{Ext}^1_{\mathcal{C}}} \\ \cup\!| & & \cup\!| \\ \boxed{\text{substructure}} & \longleftrightarrow & \boxed{\text{closed subfunctor}} \end{array}$$

At first, we introduce the notion of closed subfunctors in extriangulated categories. In the rest, $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ denotes an extriangulated category.

$\mathbb{F}$: an additive subfunctor of $\mathbb{E}$
Define $\mathfrak{s}|_{\mathbb{F}}$ by $\mathfrak{s}|_{\mathbb{F}}(\delta) := \mathfrak{s}(\delta)$ for any $\mathbb{F}$-extension δ.
Then $(\mathcal{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ satisfies $(\mathrm{ET1})$, $(\mathrm{ET2})$, $(\mathrm{ET3})$ and $(\mathrm{ET3})^{\mathrm{op}}$.

$(\mathrm{ET4})$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "d"] & {} \\
A \arrow[r, "gf"'] & C \arrow[r, "h'"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, dashed, "\delta'"'] \arrow[d, "e"] & {} \\
& F \arrow[r, equal] \arrow[d, dashed, "\gamma"] & F \arrow[r, dashed, "\gamma"'] \arrow[d, dashed, "f'\gamma"] & {} \\
& {} & {} &
\end{tikzcd}
$$

Proposition (Herschend-Liu-Nakaoka)

TFAE

1. $(\mathcal{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ *is an extriangulated category.*
2. $\mathfrak{s}|_{\mathbb{F}}$*-inflations are closed under compositions.*
3. $\mathfrak{s}|_{\mathbb{F}}$*-deflations are closed under compositions.*

We call $\mathbb{F}$ a closed subfunctor if it satisfies the above conditions. We abbreviate $(\mathcal{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ to $(\mathcal{C}, \mathbb{F})$.

Example (Dräxler-Reiten-Smalø-Solberg-Keller, Herschend-Liu-Nakaoka)

$\mathcal{D} \subseteq \mathcal{C}$: an additive subcategory. Define

$$\mathbb{E}_{\mathcal{D}}(C, A) := \{\delta \in \mathbb{E}(C, A) \mid (\delta_\sharp)_D = 0 \text{ for any } D \in \mathcal{D}\}.$$

Then $\mathbb{E}_{\mathcal{D}}$ is a closed subfunctor.

$$\mathcal{C}(D, A) \overset{f\circ-}{\to} \mathcal{C}(D, B) \overset{g\circ-}{\to} \mathcal{C}(D, C) \overset{(\delta_\sharp)_D}{\to} \mathbb{E}(D, A)$$

$$(A \overset{f}{\to} B \overset{g}{\to} C \overset{\delta}{\dashrightarrow} \text{: an } \mathfrak{s}\text{-triangle})$$

In $(\mathcal{C}, \mathbb{E}_{\mathcal{D}})$, the subcategory $\mathcal{D}$ consists of projective objects.

Theorem (Dräxler-Reiten-Smalø-Solberg-Keller)

Λ: *an artin algebra*, mod Λ: *the category of all* Λ*-modules*
Suppose that Λ *is of finite representation type.*
Then every closed subfunctor of $\mathbb{E} = \mathrm{Ext}^1_\Lambda$ *is of the form* $\mathbb{E}_{\mathcal{D}}$ *for some subcategory* $\mathcal{D} \subseteq \operatorname{mod} \Lambda$.

Aim of this section

- H : a half exact functor $\rightsquigarrow \mathbb{E}_R^H$: a closed subfunctor
- Observe the relation between half exact functors, closed subfunctors and defects of $\mathbb{E}$-extensions as the following diagram.

$$\begin{tikzcd}
\{H \mid \text{a half exact functor}\} \arrow[d] \arrow[dr, "\mathbb{E}_R^{(-)}"] \arrow[dr, phantom, "\circlearrowright" near start] & \\
\{\mathcal{S} \mid \text{a Serre subcat. of } \operatorname{def}\mathbb{E}\} \arrow[r, leftrightarrow] & \{\mathbb{F} \mid \text{a closed subfunctor of } \mathbb{E}\}
\end{tikzcd}$$

Definition (Ogawa, Liu-Nakaoka)

$\mathcal{A}$: an abelian category, $H\colon \mathcal{C} \to \mathcal{A}$: an additive functor
H: a half exact functor $:\Leftrightarrow$ For any $\mathfrak{s}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$, the sequence $HA \xrightarrow{Hf} HB \xrightarrow{Hg} HC$ is exact at HB.

Example

- Hom, $\otimes$,Ext, Tor
- half exact functors $=$ (co)homological functors where $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is triangulated.

Definition

$H\colon \mathcal{C} \to \mathcal{A}$: a half exact functor. Define a subset $\mathbb{E}_R^H(C,A) \subseteq \mathbb{E}(C,A)$ by

$$\delta \in \mathbb{E}_R^H(C,A) \quad :\Leftrightarrow \text{ For any } A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow},$$
$$HA \xrightarrow{Hf} HB \xrightarrow{Hg} HC \to 0\colon \text{ exact in } \mathcal{A}.$$

Proposition (S)

1. $\mathbb{E}_R^H$ *is a closed subfunctor of* $\mathbb{E}$.
2. $H\colon (\mathcal{C}, \mathbb{E}_R^H) \to \mathcal{A}$ *is right exact.*

Example (Herschend-Liu-Nakaoka)

$\mathcal{D} \subseteq \mathcal{C}$: a subcat. $H : \mathcal{C} \to \operatorname{Mod} \mathcal{D}; C \mapsto \mathcal{C}(-, C)|_{\mathcal{D}}$.
Then $\mathbb{E}_R^H = \mathbb{E}_{\mathcal{D}}$ holds.

Example (Krause)

$(\mathcal{C}, \mathbb{E}, \mathfrak{s})$: a compctly generated triang. cat. $\mathcal{D}$: the subcategory consisting of all compact objects. $H : \mathcal{C} \to \operatorname{Mod} \mathcal{D}; C \mapsto \mathcal{C}(-, C)|_{\mathcal{D}}$.
For an $\mathfrak{s}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$,

$$\delta \in \mathbb{E}_R^H(C, A) \Leftrightarrow A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow} : \text{pure-exact.}$$

Dually, we consider a closed subfunctor $\mathbb{E}_L^H$ of $\mathbb{E}$ which makes H a left exact functor.

Example

$H\colon \mathcal{C} \to \mathsf{Mod}\,\mathcal{C}; C \mapsto \mathcal{C}(-,C), G\colon \mathcal{C} \to \mathsf{Mod}\,\mathcal{C}^{\mathrm{op}}; C \mapsto \mathcal{C}(C,-)$.

1. $(\mathcal{C}, \mathbb{E}_L^H \cap \mathbb{E}_L^G)$ is an exact category.
2. For any closed subfunctor $\mathbb{F}$ of $\mathbb{E}$,

$$(\mathcal{C}, \mathbb{F}) \text{ is an exact category} \Rightarrow \mathbb{F} \subseteq \mathbb{E}_L^H \cap \mathbb{E}_R^G$$

Example (Cohn)

Λ: a ring, $\delta\colon 0 \to A \to B \to C \to 0$: an short exact sequence in $\mathsf{Mod}\,\Lambda$,
$M \in \mathsf{Mod}\,\Lambda^{\mathrm{op}}$: a left Λ-module, $T_M := - \otimes_\Lambda M\colon\ \mathsf{Mod}\,\Lambda \to Ab$.
Then

$$\delta \in \bigcap_{M \in \mathsf{Mod}\,\Lambda^{\mathrm{op}}} \mathbb{E}_L^{T_M}(C, A) \Leftrightarrow \delta\colon \text{pure exact}$$

Hence the class of all pure exact sequences defines a closed subfunctor $\bigcap_{M \in \mathsf{Mod}\,\Lambda^{\mathrm{op}}} \mathbb{E}_L^{T_M}$ of Ext^1_Λ.

Now we observe the relation between closed subfunctors and defects of $\mathbb{E}$-extensions.

$\operatorname{mod}\mathcal{C} \subseteq \operatorname{Mod}\mathcal{C}$: the subcat. consisting of finitely presented $\mathcal{C}$-modules F, namely, there is an exact sequence $\mathcal{C}(-,C) \to \mathcal{C}(-,D) \to F \to 0$.

Definition (Ogawa)

$\delta \in \mathbb{E}(C,A)$: an $\mathbb{E}$-extension. Define $\delta^* \in \operatorname{mod}\mathcal{C}$ by

$$\mathcal{C}(-,A) \overset{f\circ -}{\to} \mathcal{C}(-,B) \overset{g\circ -}{\to} \mathcal{C}(-,C) \to \delta_* \to 0$$

where $A \overset{f}{\to} B \overset{g}{\to} C \overset{\delta}{\dashrightarrow}$ is an $\mathfrak{s}$-triangle.
We call δ^* a (contravariant) defect of δ.

$\operatorname{def}\mathbb{E} \subseteq \operatorname{mod}\mathcal{C}$: the subcategory consisting of $\mathcal{C}$-modules isomorphic to defects.

Theorem (Enomoto)

1. $\operatorname{def}\mathbb{E}$: *an abelian category* ($\operatorname{def}\mathbb{E} \subseteq \operatorname{coh}\mathcal{C}$: *a Serre subcategory*)
2. *There are poset isomorphisms:*

$$\begin{array}{rcl} \{\mathcal{S} \mid \textit{a Serre subcat. of } \operatorname{def}\mathbb{E}\} & \leftrightarrow & \{\mathbb{F} \mid \textit{a closed subfunctor of } \mathbb{E}\} \\ \mathcal{S} & \mapsto & \mathbb{F}(\mathcal{S}) := \{\delta \in \mathbb{E} \mid \delta^* \in \mathcal{S}\} \\ \operatorname{def}\mathbb{F} & \leftarrow\!\shortmid & \mathbb{F} \end{array}$$

where $\operatorname{def}\mathbb{F} := \{F \in \operatorname{def}\mathbb{E} \mid F \cong \delta^* \textit{ with } \delta \in \mathbb{F}\}$.

Now we discuss the relation between $\mathbb{E}_R^H$ and the previous theorem. We want to construct the map depicted as the dashed arrow in the following diagram.

$$\begin{tikzcd}
\{H \mid \text{a half exact functor}\} \arrow[d, dashed, "???"'] \arrow[dr, "\mathbb{E}_R^{(-)}"] \arrow[dr, phantom, "\circlearrowright", shift right=4ex] & \\
\{\mathcal{S} \mid \text{a Serre subcat. of } \operatorname{def}\mathbb{E}\} \arrow[r, leftrightarrow, "\text{[Enomoto]}"'] & \{\mathbb{F} \mid \text{a closed subfunctor of } \mathbb{E}\}
\end{tikzcd}$$

Lemma (Krause)

$\mathcal{C}$: *an additive category*
For any additive functor $H : \mathcal{C} \to \mathcal{A}$ *with an additive category* $\mathcal{A}$ *which has cokernels, there exists, uniquely up to a natural isomorphism, a right exact functor* $\widetilde{H} : \operatorname{mod} \mathcal{C} \to \mathcal{A}$ *which makes the following diagram commute up to a natural isomorphism.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\textit{Yoneda}"] \arrow[d, "H"'] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{mod} \mathcal{C} \arrow[dl, dashed, "\widetilde{H}"] \\ \mathcal{A} & \end{tikzcd}$$

$$\mathcal{C}(-,C) \xrightarrow{f} \mathcal{C}(-,D) \to F \to 0$$
$$\xmapsto{\widetilde{H}} HC \xrightarrow{Hf} HD \to \widetilde{H}F \to 0$$

$H\colon \mathcal{C} \to \mathcal{A}$: a half exact functor.
By the previous lemma, we obtain a right exact functor $\widetilde{H}\colon \operatorname{mod}\mathcal{C} \to \mathcal{A}$.

Proposition (S)

$\widetilde{H}$ restricts to an exact functor $\widehat{H}\colon$ def $\mathbb{E} \to \mathcal{A}$.

Then we obtain a Serre subcategory $\operatorname{Ker}\widehat{H} := \{F \in \text{def}\,\mathbb{E} \mid \widehat{H}(F) = 0\}$ of def $\mathbb{E}$.

Proposition (S)

The following diagram commutes.

$$
\begin{tikzcd}
\{H \mid \textit{a half exact functor}\} \arrow[d, "\mathrm{Ker}\,\widehat{(-)}"'] \arrow[dr, "\mathbb{E}_R^{(-)}"] \arrow[dr, phantom, "\circlearrowright"' near start] & \\
\{\mathcal{S} \mid \textit{a Serre subcat. of } \mathrm{def}\,\mathbb{E}\} \arrow[r, leftrightarrow, "\textit{[Enomoto]}"'] & \{\mathbb{F} \mid \textit{a closed subfunctor of } \mathbb{E}\}
\end{tikzcd}
$$

In other words, for any half exact functor H, we have $\mathrm{Ker}\,\widehat{H} = \mathrm{def}\,\mathbb{E}_R^H$.

Proof.
For any $\mathfrak{s}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$, we obtain

$$(-,B) \xrightarrow{g\circ -} (-,C) \to \delta^* \to 0.$$

Applying $\widehat{H}$ to the above, we have

$$HB \xrightarrow{Hg} HC \to \widehat{H}\delta^* \to 0.$$

Thus $\delta \in \mathbb{E}_R^H(C,A) \Leftrightarrow \delta^* \in \operatorname{Ker}\widehat{H}$. $\square$

In the end, we give a sufficient condition where the map $\mathbb{E}_R^{(-)} \colon \{H \mid \text{a half exact functor}\} \to \{\mathbb{F} \mid \text{a closed subfunctor of } \mathbb{E}\}$ is surjective.

Theorem (S)

Assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has enough projective.
For any closed subfunctor $\mathbb{F}$ of $\mathbb{E}$, there exists a half exact functor $H \colon \mathcal{C} \to (\operatorname{def} \mathbb{E} / \operatorname{def} \mathbb{F})$ and we have $\mathbb{E}_R^H = \mathbb{F}$.

Thank you for listening.

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