Koszulity in topology

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[U] Usui's talk

Notation

[U] での "grading" は dga a g とは違う. "word length"

- coeff in $\mathbb{K}$: field (of $\forall$ char)
- dga = differential graded algebra

  i.e. $A = \{A^n\}_{n\in\mathbb{Z}}$: family of $\mathbb{K}$-mod's

  - $\mu : A\otimes A \longrightarrow A$: linear map of deg 0

    $\left(\mu^n : \bigoplus_{p+q=n} A^p\otimes A^q \longrightarrow A^n\right)$

  - $d : A \longrightarrow A$: linear map of deg 1

    $(d^n : A^n \to A^{n+1})$

  s.t. $\cdot\ \mu$: unital associative

  $\cdot\ d$: derivation

  $d(xy) = (dx)\cdot y + (-1)^{|x|} x\cdot(dy)$

  degree of x

- dgc = —— $''$ —— coalgebra

  $\left(\begin{array}{l} \Delta : C \to C\otimes C,\ d : C\to C \\ \Delta d = (d\otimes id + id\otimes d)\Delta \end{array}\right)$

- $\simeq$ denotes quasi-isom

  (i.e. $f : X \longrightarrow Y$: morph of $\left\{\begin{array}{l}\text{cpx}\\ \text{dga}\\ \text{dgc}\end{array}\right.$

  s.t. $H(f) : H(X) \xrightarrow{\cong} H(Y)$: isom)

- $\cong$ denotes isom

- For a space X,

  $\left[\begin{array}{l} C_*(X) = C_*(X;\mathbb{K}) : \text{the singular chain cpx} \\ C^*(X) = C^*(X;\mathbb{K}) : \text{—— } '' \text{ —— cochain alg} \end{array}\right.$

  $\rightsquigarrow C_*(X)$: dgc, $C^*(X)$: dga

  ($\because$ diag : $X \longrightarrow X\times X$)

- spaces are assumed to be simply connected

  $\rightsquigarrow \left\{\begin{array}{l} C_0(X) = \mathbb{K},\ C_1(X) = 0 \\ C^0(X) = \mathbb{K},\ C^1(X) = 0 \end{array}\right.$

  (need "normalization")

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§1. Bar and cobar construction

Def 1.1 (bar construction)

A : augmented dga

$(\varepsilon : A \to \mathbb{K}$: dga hom)

Define BA : (coaugmented) dgc as follows:

- $\bar{A} := \mathrm{Ker}(\varepsilon : A \to \mathbb{K}) \subset A$
- $s\bar{A}$: gr. $\mathbb{K}$-mod $\quad (s\bar{A} = \bar{A}[1])$

  $(s\bar{A})^n := \bar{A}^{n+1}$

  $\rightsquigarrow s : \bar{A} \xrightarrow{\cong} s\bar{A}$: $\mathbb{K}$-linear isom of deg (-1)

  $a \longmapsto sa$

- $BA := Ts\bar{A} = \bigoplus_{n=0}^{\infty} (s\bar{A})^{\otimes n}$ as gr. coalg.

  where

  $\Delta : BA \longrightarrow BA\otimes BA$

  is defined by $\quad sa_1\otimes\cdots\otimes sa_k \in BA$

  $\Delta([sa_1|\cdots|sa_k])$

  $:= \sum_{i=0}^{k} [sa_1|\cdots|sa_i]\otimes[sa_{i+1}|\cdots|sa_k]$

- $d := d_0 + d_1 : BA \longrightarrow BA$

  $d_0[sa_1|\cdots|sa_k]$

  $:= \sum_{i=1}^{k} \pm [sa_1|\cdots|s(da_i)|\cdots|sa_k]$

  $d_1[sa_1|\cdots|sa_k]$ $\quad$ multiplication in A

  $:= \sum_{i=1}^{k-1} \pm [sa_1|\cdots|s(a_ia_{i+1})|\cdots|sa_k]$

(c.f. Definition of group homology via "standard resolution")

Thm 1.2

G: topological group (e.g. Lie group)

Then

algebraic $\xleftarrow{\text{translate}}$ topological

$$B\,C_*(G) \simeq C_*(BG)$$

singular chain cpx — classifying space of principal G-bundle

We also have its "dual"

Def 1.3 (cobar construction)

C: coaugmented dgc

($\eta: \mathbb{K} \to C$: dgc hom)

Define ΩC: (augmented) dga as follows:

- $\bar{C} := \mathrm{Ker}(\varepsilon: C \to \mathbb{K})$ ← counit of dgc C
- $s^{-1}\bar{C}$: gr. $\mathbb{K}$-mod
  $(s^{-1}\bar{C})^n := \bar{C}^{n-1}$
- $\Omega C := T s^{-1}\bar{C}$ as gr alg
- $d := d_0 + d_1 : \Omega C \to \Omega C$
  - d_0 comes from $d: A \to A$
  - d_1 —"— $\Delta: C \to C \otimes C$

Thm 1.4 [Adams 56]

X: space

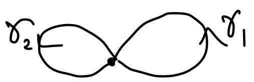
Then

$$\Omega C_*(X) \simeq C_*(\Omega X)$$

based loop space

$$\Omega X := \mathrm{Map}_*(S^1, X) = \{\gamma: S^1 \to X \mid \text{continuous base pt preserving}\}$$

$$\Omega X \times \Omega X \longrightarrow \Omega X$$ concatenation of loops



$$\pi_0(\Omega X) \cong \pi_1(X)$$

Aim

$\mathrm{Tor}^*_A(\mathbb{K},\mathbb{K}) \cong H^*(BA)$, $\mathrm{Cotor}^*_C(\mathbb{K},\mathbb{K}) \cong H^*(\Omega C)$

Compute $H^*(BA)$, $H^*(\Omega C)$

(for $C = C_*(X)$, $A = C_*(\Omega X), C^*(X)$)

Step 1 Replace A, C with smaller ones

Here we assume

- A, C: formal
  (i.e. $A \simeq H^*(A)$, $C \simeq H^*(C)$) $d = 0$
- $\begin{cases} H^*(A) \cong A(V,R) : \text{quadratic algebra} \\ H^*(C) \cong C(V,R) : \text{—"— coalgebra} \end{cases}$

Def 1.5

V: gr. $\mathbb{K}$-mod

$R \subset V^{\otimes 2}$

- $A(V,R) := TV/(R)$ ← two-sided ideal
  $$= \bigoplus_n V^{\otimes n} \Big/ \sum_{i=1}^{n-1} V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}$$
  dga with $d = 0$
- $$C(V,R) := \bigoplus_n \underbrace{\bigcap_{i=1}^{n-1} V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}}_{= K^n_n \text{ in [U, Def 1.1]}} \subset \bigoplus_n V^{\otimes n} = TV$$
  $\rightsquigarrow$ $C(V,R) \subset TV$: subcoalg.
  dgc with $d = 0$

$\rightsquigarrow \begin{cases} BA \simeq B(A(V,R)) \\ \Omega C \simeq \Omega(C(V,R)) \end{cases}$

But

$$B(A(V,R)) = B\left(TV/(R)\right) = TTV/\sim$$ ← up to shift

is still large

Step 2 Use Koszul duality!

(if $A(V,R)$, $C(V,R)$: Koszul)

§2. Twisting morphism

A : aug. dga $\quad (\mu : A \otimes A \to A)$

C : coaug. dgc $\quad (\Delta : C \to C \otimes C)$

Def 2.1

- $\tau \in \mathrm{Hom}^1(C, A)$: twisting morphism $\leftarrow$ linear map of deg 1

$\overset{\text{def}}{\Longleftrightarrow}$
- $\partial(\tau) + \tau * \tau = 0 \leftarrow$ Maurer-Cartan equation
- $\varepsilon_A \circ \tau = 0, \quad \tau \circ \eta_C = 0$

where

$\partial(\tau) := d_A \circ \tau + \tau \circ d_C$

$\tau * \tau : C \xrightarrow{\Delta} C \otimes C \xrightarrow{\tau \otimes \tau} A \otimes A \xrightarrow{\mu} A$

- $\mathrm{Tw}(C, A) := \{\tau \in \mathrm{Hom}^1(C, A) : \text{tw. morph}\}$

Def 2.2 (twisted tensor product)

M : right C-comod

N : left A-mod

$\tau \in \mathrm{Tw}(C, A)$

Define

$M \overset{\tau}{\otimes} N := M \otimes N$ as gr. mod

with differential

$d_{M \overset{\tau}{\otimes} N} := d_{M \otimes N} + d_\tau$

where

$d_{M \otimes N}(x \otimes y) := dx \otimes y + (-1)^{|x|} x \otimes dy$

$d_\tau : M \otimes N \xrightarrow{\Delta \otimes \mathrm{id}} M \otimes C \otimes N \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} M \otimes A \otimes N \xrightarrow{\mathrm{id} \otimes \mu} M \otimes N$

(e.g. $C \overset{\tau}{\otimes} A$)

Thm 2.3 [Brown 59]

$F \to E \to B$: fibration

(e.g. fiber bundle)

Then

- $\exists \tau \in \mathrm{Tw}(C_*(B), C_*(\Omega B))$ $\leftarrow$ determined only by B
- $\exists$ $C_*(\Omega B)$-mod str on $C_*(F)$

($\Omega B \curvearrowright F$: "monodromy action")

s.t.

$$\underbrace{C_*(B)}_{\text{coalg}} \overset{\tau}{\otimes} \underbrace{C_*(F)}_{C_*(\Omega B)\text{-mod}} \simeq C_*(E)$$

- We can compute $H_*(E)$ from

$C_*(B)$, $C_*(F)$, τ, $C_*(\Omega B) \curvearrowright C_*(F)$

- Chain-level refinement of Serre spectral sequence

Prop 2.4

(Assume C : conilpotent)

Define

$\tau_A : BA \longrightarrow A$

$[sa_1|\cdots|sa_k] \longmapsto \begin{cases} 0 & (k \neq 1) \\ a_1 & (k = 1) \end{cases}$

$\tau_C : C \longrightarrow \Omega C$

$1 \longmapsto 0$

$c \in \bar{C} \longmapsto [s^{-1}c]$

Then

(1) $\tau_A \in \mathrm{Tw}(BA, A)$

$\tau_C \in \mathrm{Tw}(C, \Omega C)$

(2) $\mathrm{Hom}_{\mathrm{dgc}}(C, BA)$

$\cong \big\downarrow \tau_{A*}$

$\mathrm{Tw}(C, A) \ni \tau$

$\cong \big\uparrow \tau_C^*$

$\mathrm{Hom}_{\mathrm{dga}}(\Omega C, A)$

$$\begin{tikzcd}
 & BA \arrow[dr, "\tau_A"] & \\
C \arrow[ur, dashed, "\exists!"] \arrow[rr, "\tau"] \arrow[dr, "\tau_C"'] & & A \\
 & \Omega C \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

($\hookrightarrow$ $\Omega : \mathrm{DGC} \rightleftarrows \mathrm{DGA} : B$ adjoint)

τ_A, τ_C : the universal twisting morphism

§3 Koszul duality

Notation

V: gr. $\mathbb{K}$-mod, $R \subset V^{\otimes 2}$

$$\begin{cases} A := A(V,R) = TV/(R) \\ C := C(V,R) \end{cases}$$

(V: fin. type, $V = V^{\geq 2}$ or $V^{\leq -2}$)

Def 3.1

- $A^{¡} := C(sV, s^2 R)$ ← anti-shriek

  the quadratic dual coalg of A

  (where
  $$s^2 := s^{\otimes 2} : \begin{array}{ccc} V^{\otimes 2} & \xrightarrow{\cong} & (sV)^{\otimes 2} \\ \cup & & \cup \\ R & \xrightarrow{\cong} & s^2 R \end{array}$$)

- $C^{¡} := A(s^{-1}V, s^{-2}R)$

  the quadratic dual alg of C

  (where
  $$s^{-2} := (s^{-1})^{\otimes 2} : \begin{array}{ccc} V^{\otimes 2} & \xrightarrow{\cong} & (s^{-1}V)^{\otimes 2} \\ \cup & & \cup \\ R & \xrightarrow{\cong} & s^{-2} R \end{array}$$)

⇝ $(A^{¡})^{¡} \cong A$, $(C^{¡})^{¡} \cong C$

Def 3.2

- $A^! := (A^{¡})^*$ ← linear dual

  the quadratic dual algebra of A

- $C^! := (C^{¡})^*$

  the quadratic dual coalgebra of C

Lem 3.3

(1) $A^! \cong A(s^{-1}(V^*), s^{-2}(R^{\perp}))$

(2) $C^! \cong C(s(V^*), s^2(R^{\perp}))$

(where
$$\begin{array}{ccc} V^* \otimes V^* & \xrightarrow{\cong} & (V \otimes V)^* \\ \cup & & \cup \\ \exists!\, R^{\perp} & \xrightarrow{\cong} & (V\otimes V/R)^* \end{array}$$)

Lem 3.4

We have canonical morph:

(1) $A^{¡} \overset{\iota}{\hookrightarrow} BA$: inj. hom of dgc

(2) $\Omega C \twoheadrightarrow C^{¡}$: surj. hom of dga

Proof

(1) $BA = Ts\overline{A(V,R)} = Ts(\overline{TV}/(R))$ ← bl, ← wl

⇝ BA has two "grading"

- bl: bar length
- wl: word length ← "grading" in [U]

(where $\overline{TV} := \bigoplus_{n \geq 1} V^{\otimes n}$)

(wl − bl = "syzygy degree")

wl = bl

wl \ bl	0	1 ← $sA(V,R)$	2	3
0	$\mathbb{K}$	0	0	0
1	0	sV	0	0
2	0	$s(V^{\otimes 2}/R) \xleftarrow{d}$	$(sV)^{\otimes 2}$	0
3	0	$s\left(V^{\otimes 3}/(R\otimes V + V \otimes R)\right) \xleftarrow{d}$	$s(V^{\otimes 2}/R) \otimes sV \oplus sV \otimes s(V^{\otimes 2}/R) \xleftarrow{d}$	$(sV)^{\otimes 3}$

$$\begin{aligned} A^{¡} &= C(sV, s^2R) \\ &= \bigoplus_n \bigcap_{i=1}^{n-1} (sV)^{\otimes i-1} \otimes s^2 R \otimes (sV)^{\otimes n-i-1} \\ &= \{wl = bl\} \cap \operatorname{Ker} d \end{aligned}$$

(2) $\Omega C = Ts^{-1}\overline{C(V,R)} \subset Ts^{-1}\overline{TV}$ ← cl, ← wl

wl = cl

wl \ bl	0	1	2	3
0	$\mathbb{K}$	0	0	0
1	0	$s^{-1}V$	0	0
2	0	$s^{-1}R \xrightarrow{d}$	$(s^{-1}V)^{\otimes 2}$	0
3	0	$s^{-1}(R\otimes V \cap V \otimes R) \xrightarrow{d}$	$s^{-1}R \otimes s^{-1}V \oplus s^{-1}V \otimes s^{-1}R \xrightarrow{d}$	$(s^{-1}V)^{\otimes 3}$

$$\begin{aligned} C^{¡} &= A(s^{-1}V, s^{-2}R) \\ &= \bigoplus_n (s^{-1}V)^{\otimes n} \Big/ \sum_{i=1}^{n-1} (s^{-1}V)^{\otimes i-1} \otimes s^{-2}R \otimes (s^{-1}V)^{\otimes n-i-1} \\ &= \{wl = cl\} / \operatorname{Im} d \end{aligned}$$

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By Prop 2.4,

$$\mathrm{Hom}_{dgc}(A^{i}, BA) \xrightarrow[\tau_{A*}]{\cong} \mathrm{Tw}(A^{i}, A)$$

$$L \longmapsto \quad =: \tau$$

$$\left(\begin{array}{ccccc} \tau: & A^{i} & \longrightarrow & A & \\ & \shortparallel & & \shortparallel & \\ & C(sV, s^2R) & \longrightarrow & A(V,R) & \leftarrow TV \\ TsV \ni & sv_1 \cdots sv_n & \longmapsto & \begin{cases} v_1 & (n=1) \\ 0 & (n \neq 1) \end{cases} & \ni v_1 \end{array}\right)$$

Thm 3.5

$$\boxed{\begin{array}{l} \text{TFAE:} \quad \text{Koszul complex [U. Def 11]} \\ (1)\ A^{i} \overset{\tau}{\otimes} A \simeq \mathbb{K} \\ (2)\ A^{i} \xrightarrow{\simeq} BA : \text{quasi-isom} \\ (3)\ \Omega(A^{i}) \xrightarrow{\simeq} A : \text{quasi-isom} \end{array}}$$

Applying to $A = C^{i}$,
we have similar thm for $C = C(V,R)$

Def 3.6

$$\boxed{\begin{array}{l} \cdot\ A : \underline{\text{Koszul algebra}} \\ \quad \overset{\text{def}}{\Leftrightarrow} A^{i} \xrightarrow{\simeq} BA \qquad (\Leftrightarrow (1) \Leftrightarrow (3)) \\ \cdot\ C : \underline{\text{Koszul coalgebra}} \\ \quad \overset{\text{def}}{\Leftrightarrow} \Omega C \xrightarrow{\simeq} C^{i} \qquad (\Leftrightarrow \cdots) \end{array}}$$

$$\rightsquigarrow \begin{cases} H(BA) \cong A^{i} \\ H(\Omega C) \cong C^{i} \end{cases}$$

sketch of proof of Thm 3.5

(1) $\Leftrightarrow$ (2)

In general

$$BA \overset{\tau_A}{\otimes} A \simeq \mathbb{K}$$

(cf. "standard resolution" in group homology)

Hence

$$(1) \Leftrightarrow (1')\ A^{i} \overset{\tau}{\otimes} A \xrightarrow[\simeq]{L \otimes id} BA \overset{\tau_A}{\otimes} A$$

quasi-isom

Apply "2 out of 3" to three morph's

$$\begin{cases} L : A^{i} \longrightarrow BA \\ id : A \xrightarrow{\simeq} A : \text{quasi-isom} \\ L \otimes id : A^{i} \overset{\tau}{\otimes} A \longrightarrow BA \overset{\tau_A}{\otimes} A \end{cases}$$

$\rightsquigarrow$ (2) L: quasi-isom

$\Leftrightarrow$ (1') $L \otimes id$: quasi-isom

(1) $\Leftrightarrow$ (3)

Similar argument using

$$A^{i} \overset{\tau_{A^i}}{\otimes} \Omega(A^{i}) \simeq \mathbb{K}$$

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§4. Koszul spaces

X: space ($\mathbb{K}$: field of ${}^{\forall}$char)

Def 4.1

- X: <u>formal</u> $\overset{\text{def}}{\iff} C_*(X) \simeq H_*(X)$ as dgc
  ($\iff C^*(X) \simeq H^*(X)$ as dga)
- X: <u>coformal</u> $\overset{\text{def}}{\iff} C_*(\Omega X) \simeq H_*(\Omega X)$ as dga

Thm 4.2 [Berglund '14, Berglund-Börjeson '17]

TFAE:

(1) X: formal and coformal

(2) $\left[\begin{array}{l} X: \text{formal and} \\ H^*(X): \text{Koszul alg} \end{array}\right.$

(3) $\left[\begin{array}{l} X: \text{coformal and} \\ H_*(\Omega X): \text{Koszul alg} \end{array}\right.$

Moreover, these imply

$$H_*(\Omega X) \cong (\underbrace{H^*(X)}_{\text{easy}})^! \quad \text{as alg}$$

$\rightsquigarrow$ X: <u>Koszul space</u>

<u>sketch of proof</u>

Replace (2) with its dual:

(2') $C_*(X)$: formal and $H_*(X)$: Koszul coalg

By Thm 2.3,

$\exists \tau: C_*(X) \longrightarrow C_*(\Omega X)$

s.t. $\Omega C_*(X) \xrightarrow{\simeq} C_*(\Omega X)$

$$\begin{tikzcd} \Omega C_*(X) \arrow[rr, "\simeq"] & \arrow[d, phantom, "\Omega"] & C_*(\Omega X) \\ & C_*(X) \arrow[ul, "\iota_{C_*(X)}"] \arrow[ur, "\tau"'] & \end{tikzcd}$$

<u>(1) $\Rightarrow$ (2')</u>

$$\begin{array}{ccc} \tau: C_*(X) & \longrightarrow & C_*(\Omega X) \ : \text{tw. morph} \\ \wr\! \| & & \wr\! \| \\ H_*(X) & & H_*(\Omega X) \end{array}$$

$\rightsquigarrow$ $\exists \tau': H_*(X) \longrightarrow H_*(\Omega X)$: tw. morph.

s.t. $\Omega H_*(X) \xrightarrow{\simeq} H_*(\Omega X)$

(∵ By Prop 2.4, $\mathrm{Tw}(-,-)$ is functorial w.r.t. ∞-morphisms $\Omega H_*(X) \longrightarrow \Omega C_*(X)$)

$\rightsquigarrow$ $H_*(X)$, $H_*(\Omega X)$: Koszul

(∵ $d = 0$ on $H_*(X)$, $H_*(\Omega X)$)

<u>(2') $\Rightarrow$ (1)</u>

$$\begin{array}{ccc} \tau: C_*(X) & \longrightarrow & C_*(\Omega X) \\ \wr\! \| & & \\ H_*(X) & & \end{array}$$

$\rightsquigarrow$ $\exists \tau': H_*(X) \longrightarrow C_*(\Omega X)$

s.t. $\Omega H_*(X) \xrightarrow{\simeq} C_*(\Omega X)$

Since $H_*(X)$ is Koszul,

$\Omega H_*(X) \simeq H_*(X)^{i}$

Hence

$$C_*(\Omega X) \simeq \underbrace{H_*(X)^{i}}_{d=0} \cong H_*(\Omega X)$$

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Examples 4.3

Examples of Koszul spaces:

- S^n, $\mathbb{C}P^2 \# \overset{(\ \ )}{\overline{\mathbb{C}P^2}}$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$, ...
- $\mathrm{Conf}(\mathbb{R}^n, k)$: configuration space
- "highly connected manifold"
  ($d \geq 2$, $m \leq 3d-2$
  M: $(d-1)$-connected closed mfd of dim m with $\dim H^*(X) \geq 4$)

(∵ By degree reason, no Massey product
$\rightsquigarrow$ formal
Find PBW basis of $H^*(X)$ ← Poincaré duality
$\rightsquigarrow$ $H^*(X)$: Koszul)

§5. Koszul duality of operads

$\mathcal{P}$: operad

i.e. • $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$
• $\mathcal{P}(n)$: $\mathfrak{S}_n$-mod
• $\gamma : \mathcal{P}(n)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_n) \longrightarrow \mathcal{P}(i_1+\cdots+i_n)$
"composition of operations"

⇝ $\mathcal{P}$-alg : a type of algebra determined by $\mathcal{P}$

e.g.
$\mathcal{P}$
Ass-alg = (associative algebra)
Com-alg = (commutative algebra)
Lie-alg = (Lie algebra)

$\Omega B\mathcal{P} \xrightarrow{\simeq} \mathcal{P}$: "resolution" of $\mathcal{P}$

operadic (co)bar construction

⇝ $\Omega B\mathcal{P}$-alg is "better" than $\mathcal{P}$-alg

But $\Omega B\mathcal{P}$ is too large

• $\exists$ generalization of Koszul duality to operads

e.g.
• $\mathrm{Ass}^! = \mathrm{Ass}$
• $\mathrm{Com}^! = \mathrm{Lie}$

• If $\mathcal{P}$ is Koszul,

$B\mathcal{P} \simeq \mathcal{P}^{¡}$

⇝ $\Omega B\mathcal{P} \simeq \Omega\mathcal{P}^{¡}$: small

$\Omega\mathcal{P}^{¡} = \mathcal{P}_\infty$

e.g.
Ass_∞-alg = A_∞-alg
Com_∞-alg = C_∞-alg
Lie_∞-alg = L_∞-alg