Koszul complexes, Koszul algebras, and Hilbert series

Satoshi Usui
(Tokyo Univ. of Science)

Contents

Notation
- We follow Honma's notation.
- $A = \bigoplus_{i \geq 0} A_i$: a (positively) graded ring with $A_0 = k$.
- $\otimes := \otimes_k$

§1 Koszul rings and basic properties

Def 1 A is Koszul $:\Leftrightarrow$ $\exists$ a graded projective resolution (gpr) of ${}_A k$

$\cdots \to P^{i+1} \xrightarrow{d^{i+1}} P^i \to \cdots \to P^0 \twoheadrightarrow k \cdots (\ast)$ s.t. $P^i = AP^i_i$ ($\forall i \geq 0$)

Rem 1) $P^i = AP^i_i$ implies $P^i_j = 0$ ($\forall j < i$).

2) $(\ast)$ is unique up to isom in $C(A\text{-Gr})$ and lies in $C(A\text{-Gr-}k)$.

(∵) $Q^\bullet \twoheadrightarrow {}_A k$: another gpr. Any homotopy $\varepsilon^\bullet : P^\bullet \to Q^{\bullet+1}$ is zero. ((∵) $\varepsilon^i(P^i_i) \subseteq Q^{i+1}_i = 0$). Any isom $P^\bullet \to Q^\bullet$ in $K(A\text{-Gr})$ is an isom in $C(A\text{-Gr})$. //

Ex 1) $V \in k\text{-mod}$; $T_k(V)$, $S(V)$, $\Lambda(V)$: Koszul.
((∵) $0 \to T_k(V) \otimes V \to T_k(V) \to k \to 0$: gpr)

2) $A = A_0$ is Koszul ((∵) $0 \to {}_A A \xrightarrow{id} {}_A A_0 \to 0$: gpr)

3) (Fin. dim.) quadratic monomial algebras are Koszul (Green–Huang '95)

This class contains $\Bbbk Q/R_Q^2$, $\Bbbk Q$ (Q: acyclic), gentle algs

e.g. $A : 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta$, $\beta^2 = 0$.

$${}_A A_0 = {}_A S_1 \oplus {}_A S_2, \quad P_1 = Ae_1 = \begin{smallmatrix} e_1 & 0 \\ \alpha & 1 \\ \beta\alpha & 2 \end{smallmatrix}, \quad P_2 = Ae_2 = \begin{smallmatrix} e_2 & 0 \\ \beta & 1 \end{smallmatrix} \ : \text{proj. covers of } S_i$$

$$S_1 \twoheadleftarrow P_1 \xleftarrow{-\cdot\alpha} P_2 \leftarrow 0, \quad S_2 \twoheadleftarrow P_2 \xleftarrow{-\cdot\beta} P_2 \xleftarrow{-\cdot\beta} \cdots \ : \text{min. proj. resol.}$$

$$\begin{tikzcd} P_2 & & P_2 \arrow[ll, "-\cdot\beta"'] \arrow[dl, two heads] \\ & S_2 \arrow[ul, hook] & \end{tikzcd}$$

$$A_0 \twoheadleftarrow A \xleftarrow{\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}} P_2(-1)^2 \xleftarrow{[0\ \beta]} P_2(-2) \xleftarrow{\beta} P_2(-3) \leftarrow \cdots : \text{gpr} \,/\!/$$

Observe $\forall M \in A\text{-Gr}$, $M_{\geq n} := \bigoplus_{j \geq n} M_j$ is a subobj. of M.

Lem2 $M = M_{\geq \ell} \in A\text{-Gr}$ admits a gpr ${}_A P^\bullet \twoheadrightarrow {}_A M$ with $P^i = P^i_{\geq i+\ell}$.

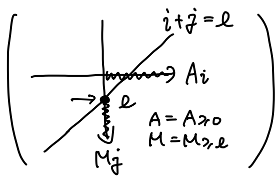
(∵) It suffices to show: $\exists P^1 \to P^0 \overset{\varepsilon}{\twoheadrightarrow} M$ s.t. $P^i = P^i_{\geq i+\ell}$: graded proj.

Consider $P^0 := A \otimes M \xrightarrow{\varepsilon} M$, $P^0 = P^0_{\geq \ell}$: graded proj

$$a \otimes m \longmapsto am$$

$\Rightarrow \operatorname{Ker}\varepsilon = \operatorname{Ker}\varepsilon_{\geq \ell} = \operatorname{Ker}\varepsilon_{\geq \ell+1}$

(∵) $f \in (\operatorname{Ker}\varepsilon)_\ell \hookrightarrow (A \otimes M)_\ell = K \otimes M_\ell \xrightarrow[\cong]{\varepsilon} M_\ell \quad \therefore f = 0$



Define $P^1 := A \otimes \operatorname{Ker}\varepsilon \longrightarrow P^0 \Rightarrow P^1 = P^1_{\geq \ell+1}$: gr. proj. //

$$\begin{tikzcd} A \otimes \operatorname{Ker}\varepsilon \arrow[rr] \arrow[dr, two heads] & \circlearrowleft & P^0 \\ & \operatorname{Ker}\varepsilon \arrow[ur, hook] & \end{tikzcd}$$

Def 3 $M \in A\text{-Gr}$ is concentrated in degree m $:\Leftrightarrow M = M_m$

(i.e. $M_i = 0$ $(\forall i \neq m)$)

Rem
- Simple objs in A-Gr are concentrated.
- Concentrated objs are semi-simple objs.

Lem 4 $\xi: 0 \to A_{\geq 1} \xrightarrow{f} A \to K \to 0$ (ex) in A-Gr

$\mathrm{ext}^1_A(K, N) = \hom_A(A_{\geq 1}, N)$ ($\forall N$: concen.)

In particular, $A_{\geq 1} = A \cdot A_1 \Leftrightarrow \mathrm{ext}^1_A(K, K(-n)) = 0$ ($\forall n \neq 1$).

(∵) $N = N_n$. ξ induces

$$\mathrm{ext}^1_A(K, N) = \mathrm{Cok}(\underbrace{\hom_A(A, N)}_{=0\ (n\neq 0)} \xrightarrow[\parallel\ 0]{-\circ f} \underbrace{\hom_A(A_{\geq 1}, N)}_{\neq 0\ (n=0)})$$

$$= \hom_A(A_{\geq 1}, N).$$

If $N = K(-n)$, then $\mathrm{ext}^1_A(K, K(-n)) = \hom_A(A_{\geq 1}, K(-n))$.

$$\Rightarrow \begin{array}{cccccccc} & & 0 & 1 & 2 & & n & \\ {}_A A_{\geq 1}: & \cdots & 0 & A_1 & A_2 & \cdots & A_n & \cdots \\ \downarrow & & & & & & \downarrow & \\ {}_A K(-n): & \cdots & 0 & 0 & 0 & \cdots & K & \cdots \end{array}$$

tells us the degrees of generators of ${}_A A_{\geq 1}$. //

Lem 5 $P^\bullet \twoheadrightarrow {}_A K$: a gpr. Assume $\exists i \geq 0$ s.t.

(i) $P^i = A P^i_i$ (ii) $\begin{array}{ccccc} Z^i & \rightarrowtail & P^i & \xrightarrow{d^i} & P^{i-1} \\ & & \uparrow & @ & \uparrow \\ (0 = Z^i_i \rightarrowtail) & & P^i_i & \rightarrowtail & P^{i-1}_i \end{array}$, $Z^i = \mathrm{Ker}\, d^i$

$\Rightarrow \mathrm{ext}^{i+1}_A(K, N) = \hom_A(Z^i, N)$ ($\forall N$: concen.)

(∵) Use $\begin{cases} Z^0 \rightarrowtail P^0 \twoheadrightarrow K \\ Z^1 \rightarrowtail P^1 \twoheadrightarrow Z^0 \\ \vdots \\ Z^i \overset{f}{\rightarrowtail} P^i \twoheadrightarrow Z^{i-1} \end{cases}$ to get

$$\begin{aligned} \mathrm{ext}^{i+1}_A(K, N) &\cong \mathrm{ext}^i_A(Z^0, N) \\ &\ \vdots \\ &\cong \mathrm{ext}^1_A(Z^{i-1}, N) \\ &\cong \mathrm{Cok}(\underbrace{\hom_A(P^i, N)}_{※\,=\,0} \xrightarrow{-\circ f} \hom_A(Z^i, N)) \\ &= \hom_A(Z^i, N). \end{aligned}$$

※ $N = N_n$

$n \neq i \Rightarrow \hom_A(P^i, N) = 0$.

$n = i \Rightarrow \begin{array}{ccc} P^i & \xrightarrow{\forall g} & N \\ f\uparrow & @ & \nearrow 0 \\ Z^i & & \end{array}$ ((∵) $P^i = AP^i_i$ and $Z^i_i = 0$)

Lem 6 TFAE for A.

1) $\mathrm{ext}_A^1(k, k(-n)) = 0 \quad (\forall n \neq 1)$

2) $A_{\geq 1} = A \cdot A_1$

3) $\exists$ a surjective graded ring homom. $T_k(A_1) \longrightarrow A$.

(Proof) (1) $\Leftrightarrow$ 2) follows from Lem 4.

2) $\Leftrightarrow$ 3) is clear //)

Prop 7 A is Koszul $\Leftrightarrow$ $\mathrm{ext}_A^i(k, k(-n)) = 0 \quad (\forall i \neq n)$

(Proof) ($\Rightarrow$) $P^\bullet \twoheadrightarrow k$: gpr with $P^i = AP_i^i$.

$\Rightarrow \mathrm{hom}_A(P^i, k(-n)) = 0 \quad (\forall i \neq n)$

$\Rightarrow \mathrm{ext}_A^i(k, k(-n)) = 0 \quad (\forall i \neq n)$.

($\Leftarrow$) By Lem 6,

$$\exists\ {}_AP^1 := A \otimes A_1 \xrightarrow{d^1} A \twoheadrightarrow {}_Ak \ (\text{ex}) \text{ s.t. } \begin{cases} P^i = AP_i^i : \text{gr. proj.} \\ d^1|_{P_1^1} : P_1^1 \to P_1^0 : \text{injective.} \end{cases}$$

(where d^1 factors as $A \otimes A_1 \twoheadrightarrow A_{\geq 1} \hookrightarrow A$, and $A = {}_AP^0$)

<u>Claim</u> $Z := \ker d^1$ satisfies $Z = AZ_2$.

$$\mathrm{hom}_A(Z, k(-n)) \overset{\text{Lem 5}}{=\!=} \mathrm{ext}_A^2(k, k(-n)) \overset{\text{assum.}}{=\!=} 0 \quad (\forall n \neq 2).$$

$\Rightarrow Z = AZ_2$, so that $P^2 := A \otimes Z_2 \xrightarrow{d^2} P^1$ satisfies (where d^2 factors as $A \otimes Z_2 \twoheadrightarrow Z \hookrightarrow P^1$)

$$\begin{cases} P^2 = AP_2^2 : \text{gr. proj.} \\ d^2|_{P_2^2} \text{ is inj.} \end{cases}$$

Repeat this argument inductively. //

Rem The above proof implies: any Koszul ring A admits a gpr of ${}_Ak$ satisfying the assumption of Lem 5 for ALL $i \geq 0$.

Cor 8 A is Koszul $\iff$ A^{op} is so.

(proof) It is enough to show ($\Rightarrow$). $\underset{\in C(A\text{-}Gr\text{-}K)}{P^\bullet} \twoheadrightarrow {}_A K$: gpr with $P^i = AP^i_i$.

$K = {}^\circledast K \hookrightarrow {}^\circledast P^0 \to {}^\circledast P^1 \to \cdots$ (ex) in A^{op}-Gr : graded inj. resol. of K_A.

$\forall i \neq n$, $\hom_{-A}(K(n), {}^\circledast P^i) = {}^*((K(n) \otimes_A P^i)_0) \cong {}^*((\underbrace{K \otimes_A P^i}_{= P^i / A_{\geq 1} \cdot P^i = P^i_i})_n) = 0.$

$\forall i \neq n$, $\operatorname{ext}^i_{-A}(K, K(-n)) = 0$

$\overset{\text{Prop 7}}{\iff}$ A^{op} : Koszul //

Lem 9 Assume $A_{\geq 1} = A \cdot A_1$

If $\operatorname{ext}^2_A(K, K(-n)) = 0$ ($\forall n \neq 2$), then A is quadratic.

Cor 10 Any Koszul ring is quadratic.

(proof) Prop 7, Lem 6, 9 //

§2 Koszul complexes and quadratic rings

Throughout, $A = T_K(V)/(R)$, $V \in K\text{-Mod-}K$, $R \subset V^{\otimes 2}$ ($\Rightarrow A_{\geq 1} = AA_1$)

Def 11 The <u>Koszul complex</u> of A is $K^\bullet \in C(A\text{-}Gr)$ given by

- $K^i = A \otimes K^i_i$, $K^i_i = \bigcap_{0 \leq l \leq i-2} V^{\otimes l} \otimes R \otimes V^{\otimes i-l-2}$ $(\subset V^{\otimes i})$
- $d^i : K^i \to K^{i-1}$, $a \otimes b_1 \otimes \cdots \otimes b_i \longmapsto ab_1 \otimes b_2 \otimes \cdots \otimes b_i$.

Rem 1) $d^i|_{K^i_i} : K^i_i \to K^{i-1}_i$ is injective ※ $\left(\begin{array}{c} d_i(1 \otimes b_1 \otimes \cdots \otimes b_i) \\ \| \\ b_1 \otimes \cdots \otimes b_i = 0 \end{array} \right)$

2) $\begin{cases} K^0_0 = K \\ K^1_1 = V \\ K^2_2 = R \end{cases} \Rightarrow \begin{cases} K^0 = A \\ K^1 = A \otimes V \\ K^2 = A \otimes R. \end{cases}$

Thm 12 A is Koszul $\iff$ $K^\bullet \twoheadrightarrow k$ is exact.

(proof) It suffices to show ($\Rightarrow$). We know: $K^1 \xrightarrow{d^1} K^0 \twoheadrightarrow k$ is exact.

$$K^1 \twoheadrightarrow A_{\geq 1} \hookrightarrow K^0$$

Claim $H^p(K^\bullet) = 0$ ($\forall p \geq 1$). Set $Z^p := \ker d^p$.

$$K^{p+1} \xrightarrow{d^{p+1}} K^p \xrightarrow{d^p} K^{p-1} \to \cdots \to K^0 \twoheadrightarrow k$$

$$K^{p+1} \twoheadrightarrow Z^p \hookrightarrow K^p \quad (\circlearrowright)$$

Assume this ($K^p \to \cdots \to K^0 \twoheadrightarrow k$) is exact.

Claim $\operatorname{Im} d^{p+1} = Z^p$.

$$\hom_A(Z^p, k(-n)) \overset{\text{Lem 5}}{=\!=} \operatorname{ext}_A^{p+1}(k, k(-n)) \overset{\text{Prop 7}}{=\!=} 0 \quad (\forall p+1 \neq n)$$

(※: $\forall w \in Z^p_{p+1}$ is of the form $a \otimes b_1 \otimes \cdots \otimes b_p$ ($a, b_i \in V$))

$Z^p = A Z^p_{p+1}$. Since $d^{p+1}(K^{p+1}_{p+1}) \overset{※}{=} Z^p_{p+1}$ and $K^{p+1} = A K^{p+1}_{p+1}$,

$\operatorname{Im} d^{p+1} = Z^p$, and thus $H^p(K^\bullet) = 0$. //

We now give another form of $K^\bullet$

Assume $A = T_k(V)/(R)$: left locally finite. ($\Rightarrow V \in k\text{-mod-}k$)

Recall $A^! := T_k(V^*)/(R^\perp)$, $R^\perp = \{ f \in (V \otimes V)^* \mid f(R) = 0 \}$.

Observe $K^\bullet$ is isom. to the following complex:

$$\cdots \to \overset{i+1}{A \otimes {}^*(A^!_{i+1})} \xrightarrow{d^{i+1}} \overset{i}{A \otimes {}^*(A^!_i)} \to \cdots \to \overset{1}{A \otimes {}^*(A^!_1)} \to \overset{0}{A} \twoheadrightarrow k.$$

$$\begin{array}{ccc} A \otimes {}^*(A^!_{i+1}) & \xrightarrow{d^{i+1}} & A \otimes {}^*(A^!_i) \\ \downarrow \cong & \circlearrowright & \uparrow \cong \\ \operatorname{Hom}_{-k}(A^!_{i+1}, A) & \longrightarrow & \operatorname{Hom}_{-k}(A^!_i, A) \end{array} \quad \text{(Identify)}$$

$$f \longmapsto [a \mapsto \sum_\alpha f(\underbrace{a \cdot \hat{v}_\alpha}_{\in A^!_{i+1}}) \cdot v_\alpha], \quad (f(a\cdot\hat{v}_\alpha) \in A)$$

where $\operatorname{Hom}_k(V, V) \cong V^* \otimes V$

$$\mathrm{id}_V \longmapsto \sum_\alpha \hat{v}_\alpha \otimes v_\alpha$$

Indeed, we have

$$^*(A^!_i) \cong \{\phi \in {}^*((V^*)^{\otimes i}) \mid \phi(\textstyle\sum_\nu X_\nu)=0\},\quad X_\nu = (V^*)^{\otimes i-\nu-2}\otimes R^\perp \otimes (V^*)^{\otimes \nu}$$

$$= \bigcap_\nu \{\phi \in {}^*((V^*)^{\otimes i}) \mid \phi(X_\nu)=0\}$$

$$\cong \bigcap_\nu \{w \in V^{\otimes i} \mid f(w)=0,\ {}^\forall f \in X_\nu\}$$

$$= \bigcap_\nu V^{\otimes \nu}\otimes R \otimes V^{\otimes i-\nu-2} = K^i_i .$$

Prop 13 A : left locally finite Koszul (LFK)

$\Rightarrow A^!$: right locally finite Koszul (RFK).

(proof) Consider $K^\bullet$ as the total "space" of the bigraded $(A, A^!)$-bimodule

$${}_A A \otimes {}^\circledast({}_{A^!}A^!) = \bigoplus_{i,j} A_i \otimes {}^\circledast(A^!)_j = \bigoplus_{i,j} A_i \otimes {}^*(A^!_{-j}) :$$

$$K_{A^!} \rightarrowtail A_0\otimes{}^\circledast(A^!) \longrightarrow A_1\otimes{}^\circledast(A^!) \longrightarrow A_2\otimes{}^\circledast(A^!) \to \cdots \text{ [ex]} \quad (\star)$$

$$\begin{array}{ccccccc}
 & \| & \| & \| & & & {}_AK \\
 & & & & & & \uparrow \\
 & A_0\otimes{}^\circledast(A^!)_0 & A_1\otimes{}^\circledast(A^!)_0 & A_2\otimes{}^\circledast(A^!)_0 & \cdots & = & A\otimes {}^*A^!_0 \\
 & & \nearrow & \nearrow & \nearrow & & \uparrow \\
 & A_0\otimes{}^\circledast(A^!)_{-1} & A_1\otimes{}^\circledast(A^!)_{-1} & A_2\otimes{}^\circledast(A^!)_{-1} & \cdots & = & A\otimes {}^*A^!_1 \\
 & & \nearrow & \nearrow & \nearrow & & \uparrow \\
 & A_0\otimes{}^\circledast(A^!)_{-2} & A_1\otimes{}^\circledast(A^!)_{-2} & A_2\otimes{}^\circledast(A^!)_{-2} & \cdots & = & A\otimes {}^*A^!_2 \\
 & \vdots & \nearrow \quad \vdots & \nearrow \quad \vdots & & & \vdots
\end{array}$$

($i \uparrow$, $\to j$)

where $d_{i,j}$ commute with the action of A and $A^!$,

$$H^{i,j}(A\otimes{}^\circledast(A^!)) = 0 \text{ for } \forall (i,j) \neq (0,0).$$

$$\begin{cases} (A_i \otimes {}^*A^!_j)^* \cong \mathrm{Hom}_K({}^*A^!_j, A^*_i) \cong A^!_j \otimes A^*_i, \\ (A_i\otimes{}^\circledast(A^!))^\circledast \cong A^!\otimes A_i \text{ in } A^!\text{-Gr}. \end{cases}$$

Applying $(\ )^\circledast$ to $(\star)$ yields a linear gpr

$${}_{A^!}K = K^\circledast \twoheadleftarrow A^!\otimes A^*_0 \longleftarrow A^!\otimes A^*_1 \leftarrow \cdots \quad //$$

§3. Quadratic dual and cohomology

R : a ring, ${}_RP^\bullet$, ${}_RQ^\bullet$: chain complexes (i.e. $\cdots \to P^i \xrightarrow{d^i} P^{i-1} \to \cdots$)

Define a cochain complex $\mathrm{Hom}_R^\bullet(P^\bullet, Q^\bullet)$ (i.e. $\cdots \to \mathrm{Hom}_R^i \xrightarrow{d^i} \mathrm{Hom}_R^{i+1} \to \cdots$) by

- $\mathrm{Hom}_R^i(P^\bullet, Q^\bullet) := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(P^n, Q^{n-i})$ $(i \in \mathbb{Z})$
- $d(f) = d_Q \circ f - (-1)^{|f|} f \circ d_P$ $(\forall f :$ homog.$)$
  $(P^n \xrightarrow{f^n} Q^{n-|f|} \xrightarrow{d_Q} Q^{n-|f|-1})$

Rem 1) $H^i \mathrm{Hom}_R^\bullet(P^\bullet, Q^\bullet) = \mathrm{Hom}_{K(R\text{-Mod})}(P^\bullet, Q^\bullet[i])$

2) $\forall f, g \in \mathrm{End}_R^\bullet(P^\bullet)$: homog., $d(f\cdot g) = d(f)\cdot g + (-1)^{|f|} f\cdot d(g)$

Here, for $\forall f = (f_n)_n$, $g = (g_n)_n$, define $f\cdot g := (f_{n-|g|} \circ g_n)_n$.

$\Rightarrow H^\bullet \mathrm{End}_R^\bullet(P^\bullet) := \bigoplus_{i\in\mathbb{Z}} H^i \mathrm{End}_R^\bullet(P^\bullet)$ becomes a graded ring

3) ${}_RP^\bullet \to {}_RM$: proj. resol.

$\Rightarrow \mathrm{Ext}_R^\bullet(M, M) \cong H^\bullet \mathrm{End}_R^\bullet(P^\bullet)$ as graded rings.

Thm 14 A : LFK, $E(A) := \mathrm{Ext}_A^\bullet(K, K)$.

(1) $A^! = E(A)^{op}$ as graded rings.

(2) $E(A)$ is LFK, and $E(E(A)) = A$.

☹ (1) Consider $\tilde{K}^\bullet$ with $\tilde{K}^i = K^i$, $d_{\tilde{K}}^i = (-1)^i d_K$. $\Rightarrow \tilde{K}^\bullet \cong K^\bullet$ in $C(A\text{-Gr})$.

Since $\tilde{K}^\bullet = A \otimes \circledast(A^!) \in (A^!)^{op}\text{-Gr}$, $\exists$ a ring homom

$(A^!)^{op} \xrightarrow{\varphi} \mathrm{End}_A^\bullet(\tilde{K}^\bullet)$, $\underset{\deg i}{a^{op}} \longmapsto \underset{\deg i}{-\cdot a}$ $(\in \prod_n \mathrm{Hom}_A(\tilde{K}^n, \tilde{K}^{n-i}))$

$$\begin{cases} d_K \circ (-\cdot a) = (-\cdot a) \circ d_K \\ d_{\tilde{K}}^i = (-1)^i d_K^i \end{cases} \Rightarrow d_{\mathrm{End}_A^\bullet(\tilde{K})}(-\cdot a) = 0.$$

∃ a ring hom $(A^!)^{op} \xrightarrow{H(\varphi)} H^\bullet \mathrm{End}^\bullet_A(\tilde{K}^\bullet) = E(A)$,

where $(A^!)^{op}$ is viewed as $[\cdots \to \overset{-1}{0} \to \overset{0}{A_0^!} \xrightarrow{0} \overset{1}{A_1^!} \xrightarrow{0} \cdots]$.

$$(A^!)^{op} \xrightarrow{\varphi} \mathrm{End}^\bullet_A(\tilde{K}) \xrightarrow{quism} \underbrace{\mathrm{Hom}^\bullet_A(\tilde{K}, k)} = A^!$$

$$= \oplus_i \underbrace{\mathrm{Hom}_A(A \otimes^* A_i^!, k)}_{\cong A_i^!}$$

<u>Observe</u> The composition is $id_{A^!}$.

⇒ φ is a quasi-isom, and hence we have $(A^!)^{op} \cong E(A)$.

(2) $E(A) = (A^!)^{op}$: LFK (☺ $V^*_k = {}_{k^{op}}V^*$)

<u>Observe</u> B : RFK ⇒ B^{op}, ${}^!B$: LFK, and $(B^{op})^! = ({}^!B)^{op}$.

$$E(E(A)) = (((\underset{\text{RFK}}{A^!})^{op})^!)^{op} = {}^!(A^!) = A. \quad //$$

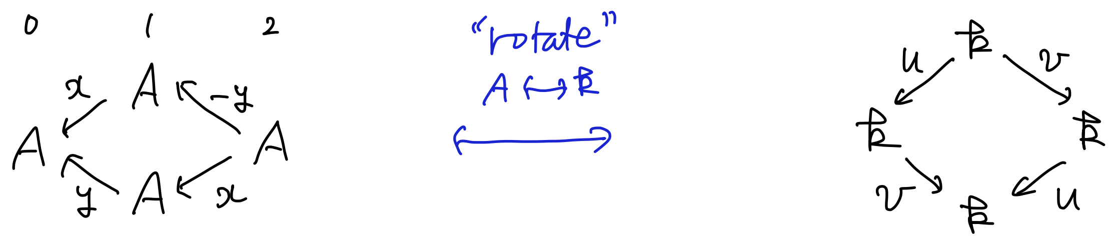
<u>Ex</u> $V = kx \oplus ky$. As will be seen,

$A = S(V) = k[x, y]$: Koszul.

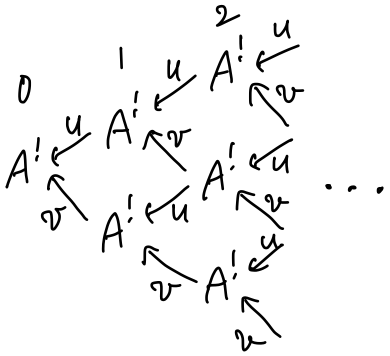
$A^! = \Lambda(V) = k\langle u, v\rangle / (u^2, v^2, uv + vu)$

Compare

① the gpr of ${}_A k$ with the (descending) Loewy str. of $A^!$



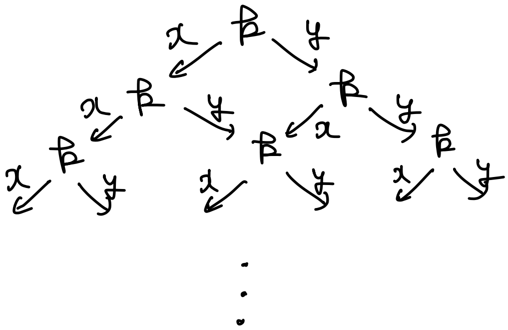
② —〃— of $A^{!}$: $\Bbbk$ with —〃— of A



"rotate"

$A^! \leftrightarrow \Bbbk$

$\longleftrightarrow$



__One may expect__

Koszul duality $\approx$ rotating gpr to get Loewy str. and vice versa.

This is not the case. The corrected statement is given in forthcoming lectures.

§4. A numerical Koszulity criterion

__Assume__ (1) $A = T_K(V)/(R)$ with $R \subset V^{\otimes 2}$ is a $\Bbbk$-alg.

(2) $\dim_{\Bbbk} A_i < \infty \quad (\forall i)$

(3) $K = \bigoplus_{x \in W} \Bbbk \cdot 1_x \quad (W : \text{a finite set}, \ 1_x \cdot 1_y = \delta_{xy} 1_x)$

__Rem__ K is comm. by (3).

__Def 15__ The __Hilbert polynomial__ of A is a $W \times W$ matix $P(A, t)$ given by

$$P(A,t)_{x,y} := \sum_{i=0}^{\infty} t^i \cdot \dim_{\Bbbk} (1_x A_i 1_y) \in \mathbb{Z}[[t]]$$

Observe A : left noetherian

$\Rightarrow E := \mathrm{Ext}_A^\bullet(k,k)$ satisfies $\begin{cases} E_0 = k \\ \dim_k E_i < \infty \quad (\forall i) \end{cases}$

$\Rightarrow$ One defines $P(E,t)$ as well.

Thm 16 A : left noeth. k-alg. satisfying (1) ~ (3).

A is Koszul $\Leftrightarrow P(A,t)\cdot P(E,-t) = E_n \quad (n = \# W)$

§5 Symmetric vs exterior algebras 2

$V = \bigoplus_{i=1}^{n} k x_i \in k\text{-mod}.$

$S^\bullet(V) = T_k(V)/(x\otimes y - y\otimes x \mid x, y V)$, $\dim_k S^\ell(V) = \binom{\ell+n-1}{\ell} < \infty$

$\Lambda^\bullet(V) = T_k(V)/(x\otimes x \mid x\in V)$, $\dim_k \Lambda^\ell(V) = \binom{n}{\ell} < \infty$

Recall $S^\bullet(V)^! = \Lambda^\bullet(V^*)$

Thm 17 $S := S^\bullet(V)$, $\Lambda := \Lambda^\bullet(V)$ are Koszul.

☺ We show : Λ is Koszul (we compute $P(E,-t)$). Define

$I_\ell := \{ \underline{m} = (m_1, \dots, m_n) \mid m_i \geq 0, \sum_{i=1}^n m_i = \ell \}$;

$\Lambda_{\underline{m}} := \Lambda$: the free Λ-module (of rank 1) gen. by $1_{\underline{m}}$;

$\underline{m}_{\hat{i}} := (m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_n) \quad (1 \leq i \leq n).$

Observe

- $\# I_\ell = \binom{\ell+n-1}{\ell}$
- $k \twoheadleftarrow \underbrace{\bigoplus_{\underline{m}\in I_0} \Lambda_{\underline{m}}}_{=\Lambda} \xleftarrow{d_1} \bigoplus_{\underline{m}\in I_1} \Lambda_{\underline{m}} \xleftarrow{d_2} \cdots - (\otimes)$, $d_i : 1_{\underline{m}} \longmapsto \sum_{\substack{i \\ m_i \neq 0}} x_i \cdot 1_{\underline{m}_{\hat{i}}}$

is a proj. resol. ${}_\Lambda P^\bullet \twoheadrightarrow {}_\Lambda k$.

//

- $\mathrm{Ext}^{\ell}_{\Lambda}(\Bbbk, \Bbbk) = H^0(\mathrm{Hom}^{\bullet}_{\Lambda}(P^{\bullet}, \Bbbk[\ell]))$

  $\cup$

  $\mathrm{Hom}_{\Lambda}(\bigoplus_{\underline{m} \in I_\ell} \Lambda_{\underline{m}}, \Bbbk) = \mathrm{span}\left\{ \Lambda_{\underline{m}} \xrightarrow{p_{\underline{m}}} \Bbbk,\ 1_{\underline{m}} \mapsto 1_{\Bbbk} \,\middle|\, \underline{m} \in I_\ell \right\}$

  $\downarrow$

  $= \mathrm{span}\{ [p_{\underline{m}}] \mid \underline{m} \in I_\ell \}$

  $\Rightarrow E = \mathrm{Ext}^{\bullet}_{\Lambda}(\Bbbk, \Bbbk)$ satisfies $\dim_{\Bbbk} E^{\ell} = \binom{\ell+n-1}{\ell}$.

- $P(\Lambda, t) = \sum_{i \geq 0} t^i \cdot \binom{n}{i} = (1+t)^n$,

  $P(E, -t) = \sum_{i \geq 0} (-t)^i \cdot \binom{\ell+n-1}{\ell} = \dfrac{1}{(1+t)^n}$

Hence, Λ is Koszul by Thm 16 //

Rem Although the proj. resol. (※) shows that Λ is Koszul, we used Thm 16 (in order to explain how to use it).