Koszulity in Combinatorial Commutative Algebra

Winter School on Koszul Algebra and Koszul Duality

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First of All

References

K. Yanagawa, Derived category of squarefree modules and local cohomology with monomial ideal support, J. Math. Soc. Japan **56** (2004), 289–308.

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Stanley–Reisner rings and Alexander dual

Stanley–Reisner rings and Alexander dual

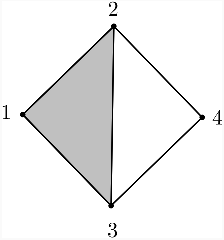
$n \in \mathbb{Z}_+ := \{k \in \mathbb{Z} \mid k > 0\}$. $[n] := \{1, 2, \ldots, n\}$.

- An (abstract) simplicial complex on $[n]$ is $\Delta \subseteq 2^{[n]}$ s.t.

$$F \subseteq G \subseteq [n],\ G \in \Delta \implies F \in \Delta.$$

- For a simpl. cpx Δ, we can construct a corresponding geometric simpl. cpx $|\Delta|$, called the geometric realization.

$$\Delta = \left\{ \begin{array}{l} \{1,2,3\}, \\ \{1,2\}, \{1,3\}, \{2,3\}, \\ \{2,4\}, \{3,4\}, \\ \{1\}, \{2\}, \{3\}, \{4\}, \varnothing \end{array} \right\}$$

$|\Delta| =$ 

Stanley–Reisner rings and Alexander dual

- $S := k[x_1, \dots, x_n]$ = a polynomial ring over a field k, considered as a $\mathbb{Z}^n$-graded algebra with $\deg x_i = e_i := (0, \dots, 0, 1, 0, \dots, 0)$.
- For $F \subseteq [n]$,

$$x_F := \begin{cases} \prod_{i \in F} x_i & F \neq \varnothing, \\ 1 & F = \varnothing, \end{cases}$$

and hence $\deg x_F = e_F := \sum_{i \in F} e_i$. Note that $F \subseteq G$ iff $x_F \mid x_G$ for $F, G \in 2^{[n]}$.

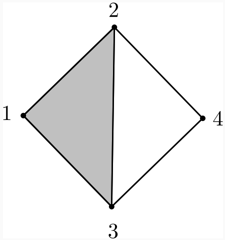
Definition

The ($\mathbb{Z}^n$-graded) ideal of S

$$\begin{aligned} I_\Delta &:= (x_F \mid F \in 2^{[n]} \setminus \Delta) \\ &= (x_F \mid F \text{ is a min. element of } 2^{[n]} \setminus \Delta) \end{aligned}$$

and ($\mathbb{Z}^n$-graded k-algebra) $k[\Delta] := S/I_\Delta$ are called the Stanley–Reisner ideal and the Stanley–Reisner ring.

Stanley–Reisner rings and Alexander dual

$|\Delta| =$ 

The minimal non-faces are $\{1, 4\}$ and $\{2, 3, 4\}$.

$$I_\Delta = (x_1x_4, x_2x_3x_4),$$
$$k[\Delta] = k[x_1, x_2, x_3, x_4]/(x_1x_4, x_2x_3x_4).$$

- $\{I_\Delta \mid \Delta \text{ is a simpl. cpx. on } \{n\}\} = \left\{\begin{matrix}\text{the ideals generated by} \\ \text{some } x_F\text{'s with } F \subseteq [n]\end{matrix}\right\}$.
- $k[\Delta]$ is designed to satisfy $k[\Delta]_F := k[\Delta]_{e_F} \neq 0$ iff $F \in \Delta$, for all $F \subseteq [n]$. Moreover as $\mathbb{Z}^n$-graded k-vector spaces,

$$k[\Delta] = \bigoplus_{F \in \Delta} k[x_i \mid i \in F]x_F,$$

and each $k[x_i \mid i \in F]x_F$ is $k[x_i \mid i \in F]$-free.

Stanley–Reisner rings and Alexander dual

Why we do a study on $k[\Delta]$

(1) Applications to enumeration of the number of the faces of $|\Delta|$ (e.g. Upper Bound Theorem and g-theorem).

(2) Interesting interaction among algebraic properties of $k[\Delta]$, combinatorial ones of Δ, and geometric ones of $|\Delta|$.

(3) A remarkable aspect of $k[\Delta]$ in view of homological algebra; e.g. a connection to Koszul duality (or BGG correspondence), detected by K. Yanagawa.

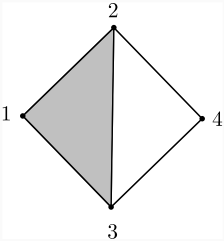
References for (1) and (2):

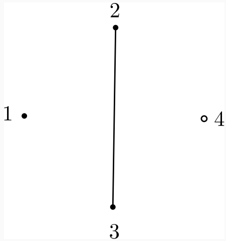
- W. Bruns and J. Herzog, Cohen–Macaulay rings, 2nd ed., Cambridge Univ. Press, 1998.
- J. Herzog and T. Hibi, Monomial Ideals, Springer, 2011.
- E. Miller and B. Sturmfels, Combinatorial Commutative Algebra, Springer, 2005.
- R. P. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, 1996.

Definition

An Alexander dual of Δ is a simplicial complex

$$\begin{aligned}\Delta^\vee &:= \{F \subseteq [n] \mid F^c := [n] \setminus F \notin \Delta\} \\ &= 2^{[n]} \setminus \{F \subseteq [n] \mid F^c \in \Delta\}\end{aligned}$$

$|\Delta| =$ 

$|\Delta^\vee| =$ 

Stanley–Reisner rings and Alexander dual

Henceforth we set $\mathbf{0} := (0, \ldots, 0) \in \mathbb{Z}^n$ and $\mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^n$.

Proposition 1

- $(\Delta^\vee)^\vee = \Delta$.
- $I_{\Delta^\vee} = (x_{F^c} \mid F \in \Delta)$.
- $(I_{\Delta^\vee})_F \cong k[\Delta]_{F^c}$ for all $F \subseteq \{n\}$.
- $\underline{\operatorname{Ext}}^i_S(k[\Delta], S(-\mathbf{1}))_F \cong \operatorname{Tor}_{\#F^c - i}(I_{\Delta^\vee}, k)_{F^c}$ for all i and $F \subseteq [n]$.
- (N. Terai around '99) $\operatorname{pd}_S(k[\Delta]) = \operatorname{reg}_S(I_{\Delta^\vee})$.
- (J. A. Eagon and V. Reiner '98) $k[\Delta]$ is Cohen–Macaulay iff $I_{\Delta^\vee}$ has a linear resolution.

The grade shift $\mathbf{1}$ is the multigraded ver. of Gorenstein parameter in the sense that

$$\underline{\operatorname{Ext}}^i_S(k, S(-\mathbf{1})) \cong \begin{cases} k & i = n = \dim S, \\ 0 & i \neq n. \end{cases}$$

Remark

Throughout the whole slides, every $\mathbb{Z}^n$-graded module M is considered as a $\mathbb{Z}$-graded one with

$$M_i := \bigoplus_{\substack{\mathbf{a} := (a_1, \ldots, a_n) \in \mathbb{Z}^n, \\ \sum_{j=1}^n a_j = n}} M_{\mathbf{a}}$$

for all i.

In the previous proposition, $\mathrm{pd}_S(-)$ and $\mathrm{reg}_S(-)$ denote projective dimension and Castelnuovo–Mumford regularity (with respect to the $\mathbb{Z}$-grading stated above).

Stanley–Reisner rings and Alexander dual

Why we call $\Delta^\vee$ "Alexander dual".

If $\Delta \subseteq \partial 2^{[n]} = 2^{[n]} \setminus \{[n]\}$, then

$$|\Delta^\vee| \underset{\text{homotopy eq.}}{\simeq} \left|\partial 2^{[n]}\right| \setminus |\Delta|.$$

The previous proposition and the celebrated Hochster's formula for local cohomology modules and Tor modules imply the following Alexander duality in the usual sense:

$$\widetilde{H}^i(|\Delta|\,;k) \cong \widetilde{H}_{(n-2)-i-1}(|\Delta^\vee|\,;k) \cong \widetilde{H}_{(n-2)-i-1}\left(\left|\partial 2^{[n]}\right| \setminus |\Delta|\,;k\right)$$

for all i.

Squarefree modules

Squarefree modules

Note that $S(-F) := S(-e_F) \cong Sx_F$ is a free left S-module for $F \subseteq [n]$.

Definition (K. Yanagawa '00)

$M \in S\text{-gr}_{\mathbb{Z}^n}$ is called squarefree iff the one of (hence all of) the following equivalent conditions:

(1)

$$\exists \bigoplus_{j=1}^{q} Sx_{G_j} \longrightarrow \bigoplus_{i=1}^{p} Sx_{F_i} \longrightarrow M \longrightarrow 0 \quad \text{(ex)}$$

where $F_i, G_j \subseteq [n]$ and $p, q \in \mathbb{Z}_+$.

(2) $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n_{\geq 0}} M_{\mathbf{a}}$ and

$$M_{\mathbf{a}} \ni m \longmapsto x_i m \in M_{\mathbf{a}+\mathbf{e}_i}$$

is k-isomorphic for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ and $i \in [n]$ with $a_i \geq 1$.

Squarefree modules

Henceforth $S\text{-Sq} := \{\text{Squarefree left } S\text{-modules}\} \underset{\text{full sub.}}{\subset} S\text{-gr}_{\mathbb{Z}^n}$.

The condition (2) says that $M \in S\text{-Sq}$ is completely determined by its squarefree part

$$M_{[\mathbf{0},\mathbf{1}]} := \bigoplus_{F \subseteq [n]} M_F.$$

Example 1

$k[\Delta]$ and I_Δ are squarefree; Indeed they have the following decomposition

$$k[\Delta] = \bigoplus_{F \in \Delta} k[x_i \mid i \in F] x_F, \quad I_\Delta = \bigoplus_{F \in 2^{[n]} \setminus \Delta} k[x_i \mid i \in F] x_F.$$

I_Δ / I_Γ is also squarefree for simpl. cpxes Δ, Γ with $\Delta \subseteq \Gamma$.

Squarefree modules

Proposition 2

(1) S-Sq is closed under extensions, Ker, Coker; In particular it is abelian. Moreover it has enough projectives and enough injectives.

(2) The indecomposable projective (resp. injective) objects are just $Sx_F \cong S(-F)$ (resp. $S/\mathfrak{p}_F = k[x_i \mid i \in F]$) with $F \subseteq [n]$, up to isom, where $\mathfrak{p}_F := (x_i \mid i \in F^c)$.

Let T denote the complex shift functor. Set

$$\omega := T^n(S(-\mathbf{1})) \in D^b(S\text{-gr}_{\mathbb{Z}^n}).$$

The localization $\omega_{\mathfrak{p}_\varnothing}$ at the graded maximal ideal $\mathfrak{p}_\varnothing = (x_1, \ldots, x_n)$ is then the normalized dualizing complex of S, and the functor

$$\mathscr{D}_S := \mathbf{R}\underline{\mathrm{Hom}}_S(-, \omega) : D^b(S\text{-gr}_{\mathbb{Z}^n}) \to D^b(\mathrm{gr}_{\mathbb{Z}^n}\text{-}S) \cong D^b(S\text{-gr}_{\mathbb{Z}^n})$$

is a duality on $D^b(S\text{-gr}_{\mathbb{Z}^n})$, and hence $\mathscr{D}_S^2 \cong \mathrm{id}_{D^b(S\text{-gr}_{\mathbb{Z}^n})}$.

Squarefree modules

Because

- A projective resolution $P^\bullet$ of $M \in S\text{-Sq}$ in $S\text{-Sq}$ consists of $Sx_F \cong S(-F)$ for some $F \in [n]$, which is also projective in $S\text{-gr}_{\mathbb{Z}^n}$,
- $\underline{\mathrm{Hom}}(S(-F), S(-\mathbf{1})) \cong S(-F^c)$ for any $F \subseteq [n]$,

we see

Proposition 3

(1) $D^b(S\text{-Sq}) \cong D^b_{S\text{-Sq}}(S\text{-gr}_{\mathbb{Z}^n}) \underset{\text{full sub.}}{\subset} D^b(S\text{-gr}_{\mathbb{Z}^n})$.

(2) $\mathscr{D}_S$ is (more precisely induces) a duality on $D^b(S\text{-Sq})$.

Squarefree modules

The cpx $\omega = T^n(S(-\mathbf{1}))$ has the following injective resolution $D^\bullet(S)$ in $D^b(S\text{-Sq})$:

$$0 \to D^{-n}(S) = S \to \cdots \to D^p(S) := \bigoplus_{\substack{F \subseteq [n], \\ \#F = -p}} S/\mathfrak{p}_F$$

$$\to \cdots \to D^0(S) = S/\mathfrak{p}_\varnothing \to 0,$$

where

$$D^p(S) \supset S/\mathfrak{p}_F \ni 1 \mapsto \sum_{i \in F} \pm 1 \in S/\mathfrak{p}_{F \cup i} \subset D^{p+1}(S).$$

In conjunction with the following (non-trivial) natural isom.

$$\underline{\mathrm{Hom}}_S(M, S/\mathfrak{p}_F)_{\geq \mathbf{0}} \cong \underline{\mathrm{Hom}}_k(M_F, k)(-F) \underline{\otimes}_k S/\mathfrak{p}_F \ (\cong (S/\mathfrak{p}_F)^{\dim_k M_F}),$$

in Sq-S for $M \in S$-Sq, we have

Squarefree modules

Proposition 4

For $M \in D^b(S\text{-Sq})$, the cpx $\mathscr{D}_S(M) \cong \mathscr{D}_S(M)_{\geq \mathbf{0}}$ is qis to the cpx $D^\bullet(M)$ defined as follows;

$$D^p(M) := \bigoplus_{\substack{i \in \mathbb{Z},\ F \subseteq [n], \\ p = -i - \#F}} \underline{\mathrm{Hom}}_k(M_F^i, k)(-F) \underline{\otimes}_k S/\mathfrak{p}_F$$

and the differential map is given as

$$f \otimes x \mapsto (-1)^p (\partial_M^i \circ f) \otimes x + f \otimes \partial_{D^\bullet(S)}^{p+i}(x)$$

for $f \otimes x \in \mathrm{Hom}_k(M_F, k)(-F) \underline{\otimes}_k S/\mathfrak{p}_F$ with $p = -i - \#F$.

Squarefree modules

- In his paper [Math. Res. Lett. **10** (2003)], Yanagawa constructed a sheaf M^+ of M on $\left|2^{[n]}\right|$ with values in k.
- The construction allowed him to generalize Hochster's formula for local cohomology modules stated above, and furthermore to detect a relation between the local duality and the Poincaré–Verdier duality through $M \to M^+$.
- See loc. cit. for details.

Generalized Alexander duality

Generalized Alexander duality

Let E be the exterior algebra of $\mathrm{Hom}_k(S_1, k)$ over k and y_i be the k-dual base of x_i. Set $\deg(y_i) := e_i$ and

$$y_F := \begin{cases} y_{i_1} \wedge y_{i_2} \wedge \cdots \wedge y_{i_s} & \text{if } F = \{i_1, \ldots, i_s\} \subseteq [n] \text{ with } i_1 < \cdots < i_s, \\ 1 & \text{if } F = \varnothing. \end{cases}$$

Definition (T. Römer '01)

$N \in E\text{-gr}_{\mathbb{Z}^n}$ is called squarefree iff it satisfies the one of (hence all of) the following equivalent conditions:

(1)

$$\exists \bigoplus_{j=1}^{q} E y_{G_j} \longrightarrow \bigoplus_{i=1}^{p} E y_{F_i} \longrightarrow N \longrightarrow 0$$

where $F_i, G_j \subseteq [n]$ and $p, q \in \mathbb{Z}_+$.

(2) $N = \bigoplus_{F \subseteq [n]} N_F$, where $N_F := N_{e_F}$.

Generalized Alexander duality

Set $E\text{-Sq} := \{\text{squarefree left } E\text{-modules}\} \underset{\text{full sub.}}{\subset} E\text{-gr}_{\mathbb{Z}^n}$.

Example 2

For a simpl. cpx Δ on $[n]$,

$$J_\Delta := E\left\langle y_F \mid F \in 2^{[n]} \setminus \Delta \right\rangle E, \quad k\langle\Delta\rangle := E/J_\Delta$$

are squarefree.

Proposition 5

E-Sq is closed under extensions, Ker, Coker; In particular it is abelian. Moreover it has enough projectives and enough injectives.

Generalized Alexander duality

- For $M \in S$-Sq with $\oplus_{j=1}^{q} Sx_{G_j} \xrightarrow{f} \bigoplus_{i=1}^{p} Sx_{F_i} \to M \to 0$, where

$$f(x_{G_j}) = \sum_{\substack{1 \le i \le p, \\ F_i \subseteq G_j}} k_{ji} x_{F_i \setminus G_j} x_{G_j}, \quad k_{ji} \in k,$$

  let M_E be the cokernel of the map $\bigoplus_{j=1}^{q} Ey_{G_j} \xrightarrow{f_E} \bigoplus_{i=1}^{p} Ey_{F_i}$ defined by

$$f_E(yy_{G_j}) = \sum_{\substack{1 \le i \le p, \\ F_i \subseteq G_j}} \pm k_{ji} yy_{G_j \setminus F_i} y_{F_i},$$

  where $\pm$ is chosen to satisfy $\pm y_{G_j \setminus F_i} y_{F_i} = y_{G_j}$.
- The module M_E is then squarefree and unique up to isom. in E-$\mathrm{gr}_{\mathbb{Z}^n}$ and assignment $M \to M_E$ gives rise to a functor $\mathcal{E} : S$-Sq $\to E$-Sq.
- Similarly we have a functor $\mathcal{S} : E$-Sq $\to S$-Sq.

Generalized Alexander duality

Theorem 1 (T. Römer, '01)
The pair of functors $(\mathcal{S}, \mathcal{E})$ is an equivalence between S-Sq and E-Sq.

- $\mathcal{E}(M)_F \cong M_F$ for all $F \subseteq [n]$ and $M \in S$-Sq;
- In particular
$$\mathcal{E}(M) \cong \bigoplus_{F \subseteq [n]} M_F$$
as $\mathbb{Z}^n$-graded k-vector spaces.
- $\mathcal{E}(k[\Delta]) \cong k\langle\Delta\rangle$ and $\mathcal{E}(I_\Delta) \cong J_\Delta$ for any simpl. cpx Δ on $[n]$.

Generalized Alexander duality

- Any $N \in \mathrm{gr}_{\mathbb{Z}^n}\text{-}E$ equipped with the structure of a left E-module with $yn = (-1)^{|y||n|}ny$ for homogeneous $y \in E$ and $n \in N$, where $|y|$ and $|n|$ denote the $\mathbb{Z}$-degrees of y and n.
- Let $\tau_E : \mathrm{gr}_{\mathbb{Z}^n}\text{-}E \to E\text{-}\mathrm{gr}_{\mathbb{Z}^n}$ be the functor induced from the observation above.
- Since E is injective, we have the duality $\mathcal{D}_E := \underline{\mathrm{Hom}}_E(-, E)$ and hence the autofunctor $\tau_E \mathcal{D}_E$ on $E\text{-}\mathrm{gr}_{\mathbb{Z}^n}$.

Theorem 2 (T. Römer '01)

(1) $\tau_E \mathcal{D}_E(N) \in E\text{-Sq}$ for $N \in E\text{-Sq}$ and hence the functor $\mathcal{A} := \mathcal{S}\tau_E\mathcal{D}_E\mathcal{E}$ is a duality on S-Sq.

(2) Furthermore $\mathcal{A}$ is a "generalized Alexander duality" in the sense that

$$\mathcal{A}(k[\Delta]) \cong I_{\Delta^\vee}$$

for any simpl. cpx Δ on $[n]$.

Generalized Alexander duality

$$\begin{array}{ccc} S\text{-Sq} & \overset{\mathcal{A}}{\dashrightarrow} & S\text{-Sq} \\ \mathcal{E}\Big\downarrow & & \Big\uparrow\mathcal{S} \\ E\text{-Sq} & \xrightarrow[\tau_E \mathcal{D}_E]{} & E\text{-Sq} \end{array}$$

Generalized Alexander duality

- Independently E. Miller also constructed a generalized Alexander duality of $M \in S\text{-Sq}$. According to his construction, $\mathcal{A}(M)$ is the squarefree module, unique up to isomorphism, satisfying

$$\mathcal{A}(M)/\mathcal{A}(M)_{>\mathbf{1}} \cong (\underline{\mathrm{Hom}}_k(M,k)(-\mathbf{1}))_{\geq \mathbf{0}}.$$

- Actually his construction is valid for positively $\mathbf{t}$-determined modules, a generalization of squarefree modules by him.

Generalized Alexander duality

Proposition 6 (E. Miller '00, T. Römer '01)

Let $M \in S$-Sq.

(1) $\mathcal{A}(M)_F \cong M_{F^c}$ for all $F \subseteq [n]$.

(2) $\operatorname{Ext}^i_S(M, S(-\mathbf{1}))_F \cong \operatorname{Tor}_{\#F^c - i}(\mathcal{A}(M), k)_{F^c}$ for all i and $F \subseteq [n]$.

(3) $\operatorname{pd}_S(M) = \operatorname{reg}_S(\mathcal{A}(M))$.

(4) M is CM iff $\mathcal{A}(M)$ has a linear resolution.

Alexander duality and Koszul duality (BGG correspondence)

Alexander duality and Koszul duality (BGG correspondence)

Proposition 7

$D^b(E\text{-Sq}) \cong D^b_{E\text{-Sq}}(E\text{-gr}_{\mathbb{Z}^n}) \underset{\text{full sub.}}{\subset} D^b(E\text{-gr}_{\mathbb{Z}^n})$.

- S and E are Koszul and $E^! = S$.
- Moreover E is of finite dimension over k and S is noetherian.
- By considering $\mathbb{Z}^n$-grading instead of $\mathbb{Z}$-grading in Koszul duality between S and E, we obtain the following.

Proposition 8

We have the equivalences $\mathscr{F} : D^b(E\text{-gr}_{\mathbb{Z}^n}) \to D^b(S\text{-gr}_{\mathbb{Z}^n})$ and $\mathscr{G} : D^b(S\text{-gr}_{\mathbb{Z}^n}) \to D^b(E\text{-gr}_{\mathbb{Z}^n})$.

Alexander duality and Koszul duality (BGG correspondence)

Let $\mathscr{S}, \mathscr{E}, \mathscr{A}$ be the functors induced from $\mathcal{S}, \mathcal{E}, \mathcal{A}$ between corresponding bounded derived categories, and $\sigma^{\mathbf{a}}$ the grade shift functor for $\mathbf{a} \in \mathbb{Z}^n$.

Theorem 3 (K. Yanagawa '04)

(1) The functor $\sigma^{-\mathbf{1}}\mathscr{F}$ induces the one $\sigma^{-\mathbf{1}}\mathscr{F} : D^b(E\text{-Sq}) \to D^b(S\text{-Sq})$.

(2) The functor $\mathscr{G}\sigma^{\mathbf{1}}$ induces the one $\mathscr{G}\sigma^{\mathbf{1}} : D^b(S\text{-Sq}) \to D^b(E\text{-Sq})$.

(3) $\sigma^{-\mathbf{1}}\mathscr{F}\mathscr{E} \cong \mathscr{A}\mathscr{D}_S$ and $\mathscr{S}\mathscr{G}\sigma^{\mathbf{1}} \cong \mathscr{D}_S\mathscr{A}$.

Alexander duality and Koszul duality (BGG correspondence)

$$
\begin{array}{ccc}
D^b(E\text{-Sq}) & \xrightarrow{\sigma^{-1}\mathscr{F}} & D^b(S\text{-Sq}) \\
\mathscr{S}\big\downarrow & & \big\| \\
D^b(S\text{-Sq}) & \xrightarrow[\mathscr{A}\mathscr{D}_S]{} & D^b(S\text{-Sq})
\end{array}
\qquad\qquad
\begin{array}{ccc}
D^b(S\text{-Sq}) & \xrightarrow{\mathscr{G}\sigma^{1}} & D^b(E\text{-Sq}) \\
\big\| & & \big\uparrow\mathscr{E} \\
D^b(S\text{-Sq}) & \xrightarrow[\mathscr{D}_S\mathscr{A}]{} & D^b(S\text{-Sq})
\end{array}
$$

Alexander duality and Koszul duality (BGG correspondence)

Sketch of proof

Let $M \in D^b(S\text{-Sq})$ and set $N := \mathscr{E}(M)$. By Prop. 4,

$$\mathscr{D}_S(M)^p = \bigoplus_{\substack{F \subseteq [n], \\ p = i - \#F}} \underline{\mathrm{Hom}}_k(M^i_F, k)(-F) \underline{\otimes}_k S/\mathfrak{p}_F$$

$$\cong \bigoplus_{\substack{F \subseteq [n], \\ p = -i - \#F}} S/\mathfrak{p}_F^{\dim_k M^i_F}.$$

Because $\mathcal{A}(S/\mathfrak{p}_F) \cong Sx_{F^c} \cong S(-F^c)$, it follows that

$$\sigma^{\mathbf{1}} \mathscr{A} \mathscr{D}_S(M)^p \cong \left(\bigoplus_{\substack{F \subseteq [n], \\ p = i + \#F}} S(-F^c) \underline{\otimes}_k M^i_F(F) \right) (\mathbf{1})$$

$$\cong \bigoplus_{\substack{F \subseteq [n], \\ p = i + \#F}} S(F) \underline{\otimes}_k M^i_F(F).$$

Alexander duality and Koszul duality (BGG correspondence)

Since $N^i \in E$-Sq and $M^i_F \cong N^i_F$ for all $F \subseteq [n]$,

$$\begin{aligned}
\mathscr{F}\mathscr{E}(M)^p &= \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} \left(\bigoplus_{p=i+|\mathbf{b}|} S_{\mathbf{a}+\mathbf{b}} \otimes_k N^i_{\mathbf{b}} \right) \\
&= \bigoplus_{p=i+|\mathbf{b}|} S(\mathbf{b}) \underline{\otimes}_k N^i_{\mathbf{b}}(\mathbf{b}) \\
&= \bigoplus_{\substack{F\subseteq[n],\\ p=i+|F|}} S(F) \underline{\otimes}_k N^i_F(F) \\
&\cong \bigoplus_{\substack{F\subseteq[n],\\ p=i+|F|}} S(F) \underline{\otimes}_k M^i_F(F) = \sigma^{\mathbf{1}} \mathscr{A}\mathscr{D}_S(M)^p.
\end{aligned}$$

Relation to Calabi–Yau property

Relation to Calabi–Yau property

Recall that T denote the translation functor.

Theorem 4 (K. Yanagawa '04)
$(\mathscr{A}\mathscr{D}_S)^3 \cong T^{-2n}$.

For example, because

$$\mathscr{A}(S(-F)) \cong S/\mathfrak{p}_{F^c}, \qquad \mathscr{A}(S/\mathfrak{p}_F(-F)) \cong S/\mathfrak{p}_{F^c}(-F^c),$$
$$\mathscr{D}_S(S(-F)) \cong T^n(S(-F^c)), \qquad \mathscr{D}_S(S/\mathfrak{p}_F) \cong T^{\#F}(S/\mathfrak{p}_F(-F)),$$
$$\mathscr{D}_S(S/\mathfrak{p}_F(-F)) \cong T^{\#F}(S/\mathfrak{p}_F)$$

for $F \subseteq [n]$, it follows that

Relation to Calabi–Yau property

$$\begin{aligned}
(\mathscr{A}\mathscr{D}_S)^3(S(-F)) &\cong (\mathscr{A}\mathscr{D})^2\mathscr{A}(T^n(S(-F^c))) \\
&\cong T^{-n}(\mathscr{A}\mathscr{D})^2(S/\mathfrak{p}_F) \\
&\cong T^{-n}(\mathscr{A}\mathscr{D})\mathscr{A}(T^{\#F}S/\mathfrak{p}_F(-F)) \\
&\cong T^{-n-\#F}\mathscr{A}\mathscr{D}(S/\mathfrak{p}_{F^c}(-F^c)) \\
&\cong T^{-n-\#F}\mathscr{A}(T^{\#F^c}(S/\mathfrak{p}_{F^c})) \\
&\cong T^{-n-\#F-\#F^c}\mathscr{A}(S/\mathfrak{p}_{F^c}) \cong T^{-2n}(S(-F)),
\end{aligned}$$

for all $F \subseteq [n]$.

- Actually, S-Sq is equivalent to the category of left modules over the tensor of n-copies of the path algebra of the quiver of type A_2.
- As a result, The natural isomorphism $(\mathscr{A}\mathscr{D}_S)^3 \cong T^{-2n}$ can be deduced from the fact that the category above is fractionally Calabi–Yau of dimension $n/3$ (Suggestion of O. Iyama to Yanagawa around 2007).

Relation to Calabi–Yau property

Let Λ_d be the path algebra of the following quiver of type A_{d+1}.

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow d$$

- Note that the k-basis of Λ_1 consists of $e_{0,0} := e_0, e_{1,1} := e_1, e_{1,0}$.
- Set $\Lambda := \Lambda_1^{\otimes_k n}$ and

$$e_{GF} := e_{\chi_G(1),\chi_F(1)} \otimes_k \cdots \otimes_k e_{\chi_G(n),\chi_F(n)}$$

  for $F \subseteq G \subseteq [n]$, where χ_F, χ_G denote the characteristic function.
- The k-basis of Λ then consists of all the e_{GF} with $F \subseteq G \subseteq [n]$, and

$$e_{IH} e_{GF} = \begin{cases} 0 & G \neq H, \\ e_{IF} & H = G \end{cases} \quad \text{for } F \subseteq G \subseteq [n] \text{ and } H \subseteq I \subseteq [n].$$

Relation to Calabi–Yau property

- For $M \in S$-Sq, the squarefree part $M_{[\mathbf{0},\mathbf{1}]} := \bigoplus_{F \subseteq [n]} M_F$ has the structure of a left Λ-module with the scalar multiplication

$$e_{GF}\, m = \begin{cases} x_{G \setminus F}\, m & F = H, \\ 0 & F \neq H \end{cases} \quad (F \subseteq G \subseteq [n]\,,\ H \subseteq [n]\,,\ m \in M_H).$$

- Moreover the k-linear map $M_{[\mathbf{0},\mathbf{1}]} \to N_{[\mathbf{0},\mathbf{1}]}$ induced from a morphism $M \to N$ in S-Sq is then Λ-linear.
- Thus we have the functor $\Phi_\Lambda : S\text{-Sq} \to \Lambda\text{-mod}$.

Proposition 9 (K. Yanagawa '04)

The functor Φ_Λ is equivalence.

Relation to Calabi–Yau property

Example 3

$\Phi_\Lambda(I_\Delta) = \Lambda\{e_{F\varnothing} \mid F \notin \Delta\}$ for a simpl. cpx Δ on $[n]$; In particular

$$\Phi_\Lambda(Sx_F) \cong \Lambda e_F, \quad \Phi_\Lambda(k[\Delta]) \cong \Lambda e_\varnothing / \Lambda\{e_{F\varnothing} \mid F \in \Delta\},$$

where $e_G = e_{GG}$ for $G \subseteq [n]$.

- Let Φ_S be the inverse of Φ_Λ. The functor between $D^b(S\text{-Sq})$ and $D^b(\Lambda\text{-mod})$ induced from Φ_S, Φ_Λ are also denoted by them.
- Set $\mathscr{D}_k := \mathbf{R}\mathrm{Hom}_k(-, k)$ and $\mathscr{D}_\Lambda := \mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda)$.
- Let $\tau_\Lambda : D^b(\text{mod-}\Lambda) \to D^b(\Lambda\text{-mod})$ be the equivalence induced from the ring isomorphism $\Lambda \ni e_{GF} \mapsto e_{F^cG^c} \in \Lambda^{\mathrm{op}}$, where Λ^{op} denote the opposite ring of Λ.

Relation to Calabi–Yau property

Theorem 5 (K. Yanagawa '04)

$\Phi_S \tau_\Lambda \mathscr{D}_k \Phi_\Lambda \cong \mathscr{A}$ and $\Phi_S \tau_\Lambda \mathscr{D}_\Lambda \Phi_\Lambda \cong T^{-n} \mathscr{D}_S$.

$$\begin{array}{ccc} D^b(S\text{-Sq}) & \xrightarrow{\mathscr{A}} & D^b(S\text{-Sq}) \\ \downarrow \Phi_\Lambda & & \uparrow \Phi_S \\ D^b(\Lambda\text{-mod}) & \xrightarrow{\tau_\Lambda \mathscr{D}_k} & D^b(\Lambda\text{-mod}) \end{array} \qquad \begin{array}{ccc} D^b(S\text{-Sq}) & \xrightarrow{T^{-n}\mathscr{D}_S} & D^b(S\text{-Sq}) \\ \downarrow \Phi_\Lambda & & \uparrow \Phi_S \\ D^b(\Lambda\text{-mod}) & \xrightarrow{\tau_\Lambda \mathscr{D}_\Lambda} & D^b(\Lambda\text{-mod}). \end{array}$$

In particular, $\mathscr{A} \mathscr{D}_S \cong T^{-n} \Phi_S \mathscr{D}_k \mathscr{D}_\Lambda \Phi_\Lambda$.

Relation to Calabi–Yau property

Let $\mathcal{T}$ be a k-linear triangulated category with $\dim_k \operatorname{Hom}_{\mathcal{T}}(X, Y) < \infty$ for all $X, Y \in \mathcal{T}$. Let n, d be positive integers.

- A k-linear autofunctor F on $\mathcal{T}$ is said to be a Serre functor if there exists a k-linear isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(Y, F(X)) \cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathcal{T}}(X, Y))$$

  functorial in $X, Y \in \mathcal{T}$ for all $X, Y \in \mathcal{T}$.
- $\mathcal{T}$ is said to be fractionally Calabi–Yau of dimension n/d (abbrev. n/d-CY) if it has a Serre functor and there exists an isomorphism of k-linear functors $F^d \cong T^n$.

Relation to Calabi–Yau property

Proposition 10 (D. Happel)

For a finite-dimensional k-algebra A of finite global dimension,

$$\mathscr{D}_k \mathscr{D}_A \cong \mathscr{D}_k(A) \overset{\mathbf{L}}{\otimes}_A - : D^b(A\text{-mod}) \to D^b(A\text{-mod})$$

is a Serre functor, where $\mathscr{D}_A := \mathbf{R}\mathrm{Hom}_A(-, A)$.

Proposition 11 (M. Herschend and O. Iyama '11)

Let A_i ($i = 1, 2$) be finite-dimensional k-algebra of finite global dimension. Assume $A_1 \otimes_k A_2$ is also of finite global dimension, and $D^b(A_1\text{-mod})$ (resp. $D^b(A_2\text{-mod})$) is m_1/l_1-CY (resp. m_2/l_2-CY). Then $D^b(A_1 \otimes_k A_2\text{-mod})$ is m/l-CY, where $l = \mathrm{lcm}(l_1, l_2)$ and $m = l((m_1 l_2 + l_1 m_2)/(l_1 l_2))$.

Relation to Calabi–Yau property

Proposition 12 (M. Kontsevich and E. Kreines)

The category $D^b(\Lambda_d\text{-mod})$ is Calabi–Yau of dimension $d/(d+2)$.

- Since Λ_d is finite-dimensional k-algebra of finite global dimension, the functor $\mathscr{D}_k\mathscr{D}_{\Lambda_d}$ is a Serre functor.
- $\Lambda = \Lambda_1^{\otimes_k n}$ is of finite global dimension, since $\Lambda\text{-mod} \cong S\text{-Sq}$.
- Consequently, we see that $\mathscr{D}_k\mathscr{D}_\Lambda$ is also a Serre functor and $D^b(\Lambda\text{-mod})$ is Calabi–Yau of dimension $n/3$.

Corollary 1 (O. Iyama around '07)

The isomorphism $(\mathscr{A}\mathscr{D}_S)^3 \cong T^{-2n}$ is deduced from the fact that $D^b(\text{mod-}\Lambda)$ is Calabi–Yau of dimension $n/3$.

Indeed

$$(\mathscr{A}\mathscr{D}_S)^3 \cong T^{-3n}\Phi_S(\mathscr{D}_k\mathscr{D}_\Lambda)^3\Phi_\Lambda \cong T^{-2n}.$$

Relation to Calabi–Yau property

Remark

According to the paper [Compos. Math. **129** (2001)] of J. Miyachi and A. Yekutieli, Proposition 12 was already proved by E. Kreines and had been known also to M. Kontsevich.

Further developments

Further developments

Let $\mathbf{t} := (t_1, \ldots, t_n) \in \mathbb{Z}^n$ with $t_i \geq 1$ for all i, and define

$$\mathbf{a} := (a_1, \ldots, a_n) \leq \mathbf{b} := (b_1, \ldots, b_n) \iff \forall i,\ a_i \leq b_i.$$

Definition (E. Miller '00)

$M \in S\text{-gr}_{\mathbb{Z}^n}$ is said to be positively **t**-determined if it satisfies one of (hence all of) the following equivalent conditions:

(1) $\exists \bigoplus_{j=1}^{q} S(-\mathbf{b}_j) \longrightarrow \bigoplus_{i=1}^{p} S(-\mathbf{a}_i) \longrightarrow M \longrightarrow 0$ with $\mathbf{0} \leq \mathbf{a}_i \leq \mathbf{t}$ and $\mathbf{0} \leq \mathbf{b}_j \leq \mathbf{t}$ for all i, j.

(2) $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n_{\geq 0}} M_{\mathbf{a}}$ and the map

$$M_{\mathbf{a}} \ni m \mapsto x_i m \in M_{\mathbf{a}+e_i}$$

is isomorphic for $\mathbf{a} \in \mathbb{Z}^n$ and i with $a_i \geq t_i$.

Further developments

- A pos. $\mathbf{1}$-det. module is just a squarefree module.
- Miller defined the Alexander duality functor $\mathcal{A}_{\mathbf{t}}$ on the category $S\text{-Sq}_{\mathbf{t}} = \{\text{pos. } \mathbf{t}\text{-det. modules}\} \underset{\text{full sub.}}{\subset} S\text{-gr}_{\mathbb{Z}^n}$.
- $\mathscr{D}_{\mathbf{t}} := \mathbf{R}\underline{\mathrm{Hom}}_S(-, T^n S(-\mathbf{t}))$ is a duality on $S\text{-Sq}_{\mathbf{t}}$.
- $\mathscr{A}_{\mathbf{1}} = \mathscr{A}$ and $\mathscr{D}_{\mathbf{1}} = \mathscr{D}_S$.
- Most of results stated in this lecture can be generalized to pos. $\mathbf{t}$-det. modules.
- See Miller's paper [J. Algebra **231** (2000)] for details of basic properties,
- and the one [Adv. Math. **226** (2011)] by M. Brun and G. Fløystad for the functor $\mathscr{A}_{\mathbf{t}}\mathscr{D}_{\mathbf{t}}$.

Thank you for your attention.