Koszul duality functors

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Winter school on Koszul algebra and Koszul duality

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Setup $k = A_0$: semisimple , $A = \bigoplus_{i \geq 0} A_i$

A-Gr : the category of ($\mathbb{Z}$-)graded left A-modules

$C(A)$: ________________ complexes of graded left A-modules

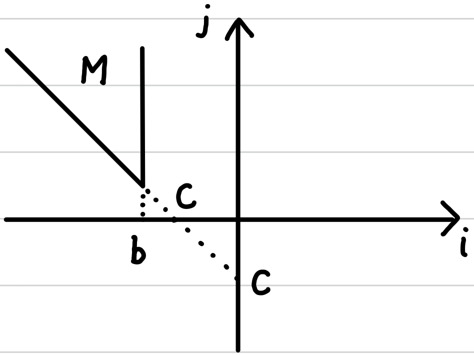
$$\begin{array}{l} \rotatebox{90}{$\in$} \\ M = (\cdots \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow \cdots) \end{array}$$

$C^{\uparrow}(A) := \{ M \in C(A) \mid M^i_j = 0 \;\; \text{if} \;\; i \gg 0 \;\text{or}\; i+j \ll 0 \}$

$C^{\downarrow}(A) := \{ M \in C(A) \mid M^i_j = 0 \;\; \text{if} \;\; i \ll 0 \;\text{or}\; i+j \gg 0 \}$

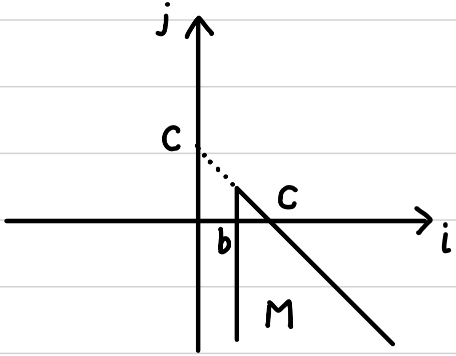
$M \in C^{\uparrow}(A) \Longleftrightarrow \exists\, b, c \in \mathbb{Z}$ s.t.

$M^i_j = 0$ if $i > b$ or $i + j < c$.



$M \in C^{\downarrow}(A) \Longleftrightarrow \exists\, b, c \in \mathbb{Z}$ s.t.

$M^i_j = 0$ if $i < b$ or $i + j > c$.



$K(A)$: the homotopy category of complexes of graded left A-modules

$K^{\uparrow}(A), K^{\downarrow}(A) \underset{\text{full sub}}{\subset} K(A)$

$\mathrm{Ob}\,K^{\uparrow}(A) := \mathrm{Ob}\,C^{\uparrow}(A) \qquad \mathrm{Ob}\,K^{\downarrow}(A) := \mathrm{Ob}\,C^{\downarrow}(A)$

$K_{ac}(A) \subset K(A)$

$\overset{\Psi}{M} \underset{\text{def}}{\Leftrightarrow} M$: acyclic, i.e., $H^i(M) = 0 \ \forall i \in \mathbb{Z}$

$K_{ac}^{\uparrow}(A) := K^{\uparrow}(A) \cap K_{ac}(A)$

$K_{ac}^{\downarrow}(A) := K^{\downarrow}(A) \cap K_{ac}(A)$

$D(A) := K(A)/K_{ac}(A)$

$D^{\uparrow}(A) := K^{\uparrow}(A)/K_{ac}^{\uparrow}(A) \qquad D^{\downarrow}(A) := K^{\downarrow}(A)/K_{ac}^{\downarrow}(A)$

Rem $D^{\uparrow}(A) \xhookrightarrow{\text{can}} D(A) \xhookleftarrow{\text{can}} D^{\downarrow}(A)$

A-Mod : the category of left A-modules

A-mod : ______________ finitely presented left A-modules

A-gr : ______________________________ graded left A-modules

Aim • A : left (locally) finite Koszul $\Rightarrow$ $D^{\downarrow}(A) \overset{\exists}{\cong} D^{\uparrow}(A^!)$.

• A : Koszul, $A \in k$-mod, $A \in k^{op}$-mod, $A^!$: left noeth.

$\Rightarrow$ $D^b(A\text{-gr}) \overset{\exists}{\cong} D^b(A^!\text{-gr})$

Henceforth, A is assumed to be left finite Koszul.

<u>Step1</u> Construct a functor $F : C^{\downarrow}(A) \longrightarrow C^{\uparrow}(A^!)$.

$M \in C^{\downarrow}(A)$

$$
\begin{aligned}
FM &:= A^! \otimes M \\
&= \bigoplus_{l,i\in\mathbb{Z}} A^!_l \otimes M^i \\
&\cong \bigoplus_{l,i\in\mathbb{Z}} (\underset{k\text{-mod}}{\underset{\cap}{{}^*(A^!_l)}})^* \otimes M^i \\
&\cong \bigoplus_{l,i\in\mathbb{Z}} \operatorname{Hom}_k({}^*(A^!_l), M^i) \\
&\cong \bigoplus_{l,i\in\mathbb{Z}} \operatorname{Hom}_k({}^*(A^!_l), \operatorname{Hom}_A(A, M^i)) \\
&\cong \bigoplus_{l,i\in\mathbb{Z}} \operatorname{Hom}_A(A \otimes {}^*(A^!_l), M^i)
\end{aligned}
$$

$\otimes = \otimes_k$

$K = \Big(\cdots \longrightarrow A \otimes {}^*(A^!_\ell) \xrightarrow{d_\ell} A \otimes {}^*(A^!_{\ell-1}) \longrightarrow \cdots \Big)$: Koszul complex

$$\begin{tikzcd}
\mathrm{Hom}_A(A \otimes {}^*(A^!_{\ell-1}), M^i) \arrow[r, "\circ d_\ell"] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Hom}_A(A \otimes {}^*(A^!_\ell), M^i) \arrow[d, phantom, sloped, "\cong"] \\
A^!_{\ell-1} \otimes M^i & A^!_\ell \otimes M^i \\
a \otimes m \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & d'(a \otimes m) := (-1)^{i+j} \Sigma\, a \check{v}_\alpha \otimes v_\alpha m \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$(m \in M^i_j)$

$$\begin{tikzcd}
\mathrm{Hom}_k(A_1, A_1) \cong (A_1)^* \otimes A_1 & \\
\mathrm{Id}_{A_1} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \Sigma\, \check{v}_\alpha \otimes v_\alpha
\end{tikzcd}$$

$$\begin{tikzcd}
A^!_\ell \otimes M^i \arrow[r] & A^!_\ell \otimes M^{i+1} \\
a \otimes m \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & d''(a \otimes m) := a \otimes \partial_M(m) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$FM = \bigoplus_{\ell,i\in\mathbb{Z}} A^!_\ell \otimes M^i$$

$$\left({}^{\exists} b,c\in\mathbb{Z} \quad \text{s.t.} \quad \begin{cases} i<b \Rightarrow M^i=0 \\ i+j>c \Rightarrow M^i_j=0 \end{cases} \qquad \because \; M\in C^{\downarrow}(A) \qquad \triangle_M \right)$$

$$(FM)^p_q := \bigoplus_{\substack{p=i+j \\ q=\ell-j}} A^!_\ell \otimes M^i_j$$

$p=b+1$

$q=1$

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & A^!_2\otimes M^b_1 \arrow[r] \arrow[u] & A^!_2\otimes M^{b+1}_1 \arrow[r] \arrow[u] & A^!_2\otimes M^{b+2}_1 \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & A^!_1\otimes M^b_0 \arrow[r] \arrow[u] & A^!_1\otimes M^{b+1}_0 \arrow[r] \arrow[u] & A^!_1\otimes M^{b+2}_0 \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & A^!_0\otimes M^b_{-1} \arrow[r] \arrow[u] & A^!_0\otimes M^{b+1}_{-1} \arrow[r] \arrow[u] & A^!_0\otimes M^{b+2}_{-1} \arrow[r] \arrow[u] & {} \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

ℓ

$d' \nearrow$

$\xrightarrow{d''}$

ℓ

$q = 0$

d' ↗

$p = b+1$

$$\begin{tikzcd}
 & & & & & {} & & {} & & {} & \\
 & & 0 \arrow[rr] & & A_2^! \otimes M_2^{b} \arrow[rr] \arrow[ur] & & A_2^! \otimes M_2^{b+1} \arrow[rr] \arrow[ur] & & A_2^! \otimes M_2^{b+2} \arrow[rr] \arrow[ur] & & {} \\
 & 0 \arrow[rr] & & A_1^! \otimes M_1^{b} \arrow[rr] \arrow[ur] & & A_1^! \otimes M_1^{b+1} \arrow[rr] \arrow[ur] & & A_1^! \otimes M_1^{b+2} \arrow[rr] \arrow[ur] & & {} & \\
0 \arrow[rr] & & A_0^! \otimes M_0^{b} \arrow[rr] \arrow[ur] & & A_0^! \otimes M_0^{b+1} \arrow[rr] \arrow[ur] & & A_0^! \otimes M_0^{b+2} \arrow[rr] \arrow[ur] & & {} & & \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & & & &
\end{tikzcd}$$

d'' →

i

ℓ

$q = -1$

d' ↗

$p = b+1$

$$\begin{tikzcd}
 & & & & & {} & & {} & & {} & \\
 & & 0 \arrow[rr] & & A_2^! \otimes M_3^{b} \arrow[rr] \arrow[ur] & & A_2^! \otimes M_3^{b+1} \arrow[rr] \arrow[ur] & & A_2^! \otimes M_3^{b+2} \arrow[rr] \arrow[ur] & & {} \\
 & 0 \arrow[rr] & & A_1^! \otimes M_2^{b} \arrow[rr] \arrow[ur] & & A_1^! \otimes M_2^{b+1} \arrow[rr] \arrow[ur] & & A_1^! \otimes M_2^{b+2} \arrow[rr] \arrow[ur] & & {} & \\
0 \arrow[rr] & & A_0^! \otimes M_1^{b} \arrow[rr] \arrow[ur] & & A_0^! \otimes M_1^{b+1} \arrow[rr] \arrow[ur] & & A_0^! \otimes M_1^{b+2} \arrow[rr] \arrow[ur] & & {} & & \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & & & &
\end{tikzcd}$$

d'' →

i

$$(FM)^p_q = \bigoplus_{\substack{p=i+j \\ q=\ell-j}} A^!_\ell \otimes M^i_j = \bigoplus_j A^!_{q+j} \otimes M^{p-j}_j$$

$$\begin{aligned}(FM)^p &= \bigoplus_q \Big(\bigoplus_j A^!_{q+j} \otimes M^{p-j}_j \Big) \\ &= \bigoplus_j \Big(\bigoplus_q A^!_{q+j} \otimes M^{p-j}_j \Big) \\ &= \bigoplus_j \Big(A^! \otimes M^{p-j}_j \Big)\end{aligned}$$

$$\begin{tikzcd}
& \bigoplus_q\left(\bigoplus_j A^!_{q+j}\otimes M_j^{p-j}\right) \arrow[d, equal] & \bigoplus_q\left(\bigoplus_j A^!_{q+j}\otimes M_j^{p+1-j}\right) \arrow[d, equal] & \\
& (FM)^p \arrow[r, "d^p"] \arrow[d, equal] & (FM)^{p+1} \arrow[d, equal] & \\
& \bigoplus_j\left(A^!\otimes M_j^{p-j}\right) & \bigoplus_j\left(A^!\otimes M_j^{p+1-j}\right) & \\
& A^!\otimes M_j^{p-j} \arrow[u, phantom, sloped, "\subset"] & \left(A^!\otimes M_{j+1}^{p-j}\right)\oplus\left(A^!\otimes M_j^{p+1-j}\right) \arrow[u, phantom, sloped, "\subset"] & \\
\text{deg. } \ell-j & a\otimes m \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & d^p(a\otimes m) := (-1)^p\sum \overbrace{a\check{v}_\alpha}^{\substack{A^!_{\ell+1}\\ \rotatebox{90}{$\in$}}}\otimes\overbrace{v_\alpha m}^{\substack{M_{j+1}\\ \rotatebox{90}{$\in$}}} \arrow[u, phantom, sloped, "\in"] & \text{deg. } \ell-j \\
& (a\in A^!_\ell) & + a\otimes\underbrace{\partial_M(m)}_{\substack{\rotatebox{-90}{$\in$}\\ M_j^{p+1-j}}} &
\end{tikzcd}$$

d^p is a homomorphism of graded left $A^!$-modules.

$$(FM)^p = \bigoplus_q \Big(\bigoplus_j A^!_{q+j} \otimes M_j^{p-j} \Big) = 0 \quad \text{if} \quad p = (p-j)+j > c$$

$$(FM)^p_q = \bigoplus_j A^!_{q+j} \otimes M_j^{p-j} = 0 \quad \text{if} \quad p+q = (p-j)+(q+j) < b$$

$$\left(\begin{array}{ll} \because & i+j > c \Rightarrow M^i_j = 0 \\ & i < b \Rightarrow M^i = 0 \\ & l < 0 \Rightarrow A^!_l = 0 \end{array} \right)$$

$$\therefore \quad FM \in C^{\uparrow}(A^!) \left(\begin{array}{l} \Longleftrightarrow \ {}^{\exists} b', c' \in \mathbb{Z} \ \text{s.t.} \\ \quad M^i = 0 \ \text{if} \ i > b', \text{ and} \\ \quad M^i_j = 0 \ \text{if} \ i+j < c'. \end{array} \right)$$

We can easily check that the assignment $M \longmapsto FM$ defines an additive functor $F : C^{\downarrow}(A) \longrightarrow C^{\uparrow}(A^!)$. $\square$

<u>Step 2</u> Prove that F induces a triangulated functor $K^{\downarrow}(A) \to K^{\uparrow}(A^!)$

Let $f : M \longrightarrow N$ be a morphism in $C^{\downarrow}(A)$.

(1) f : null-homotopic $\underset{\text{(well-known)}}{\Longleftrightarrow}$ $0 \to N \longrightarrow C(f) \longrightarrow M[1] \to 0$: split in $C(A)$.

(2)

$$\begin{tikzcd}
 & M \arrow[d, "f"] & & & \\
0 \arrow[r] & N \arrow[r] & C(f) \arrow[r] & M[1] \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & (FM)_q \arrow[d, "(Ff)_q"] & & & \\
0 \arrow[r] & (FN)_q \arrow[r] & (FC(f))_q \arrow[r] \arrow[d, equal] & (F(M[1]))_q \arrow[r] \arrow[d, equal] & 0 \\
 & & (C(Ff))_q & ((FM)[1])_q &
\end{tikzcd}$$

$\therefore$ $FC(f) = C(Ff)$ and $F(M[1]) = (FM)[1]$.

(3)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, bend right, "F"'] & N \arrow[r] & C(f) \arrow[r] & M[1] \arrow[r] & 0 \ : \text{split} \\
0 \arrow[r] & FN \arrow[r] & C(Ff) \arrow[r] & (FM)[1] \arrow[r] & 0 \ : \text{split} \iff Ff : \text{null-homotopic}
\end{tikzcd}$$

by (1).

(4) By (2) and (3) F induces a triangulated functor $K^{\downarrow}(A) \to K^{\uparrow}(A^{!})$

$\square$

<u>Step3</u> Prove that $F : K^{\downarrow}(A) \to K^{\uparrow}(A^{!})$ induces a triangulated functor $D^{\downarrow}(A) \longrightarrow D^{\uparrow}(A^{!})$.

It suffices to show that $FM \in K^{\uparrow}_{ac}(A^!)$ whenever $M \in K^{\downarrow}_{ac}(A)$

This follows from the construction of F and a standard argument of double complexes. $\square$

<u>Step 4</u> Define a functor $G : C^{\uparrow}(A^!) \longrightarrow C^{\downarrow}(A)$.

$N \in C^{\uparrow}(A^!)$ $\quad (GN)^p_q := \bigoplus_{\substack{p=i+j \\ q=\ell-j}} \mathrm{Hom}_k(A_{-\ell}, N^i_j)$

$f \in \mathrm{Hom}_k(A_{-\ell}, N^i)$ $\qquad d'(f) := (-1)^i \sum \check{v}_\alpha f(v_\alpha \cdot -)$

$$d''(f) := \partial_N(f)$$

q = 0

P = 0

$$\begin{tikzcd}
 & & & 0 & 0 & & \\
 & \sum \check{v}_\alpha f(v_\alpha \cdot -) \in & {} \arrow[r] & (A_0, N_0^0) \arrow[u] \arrow[r] & (A_0, N_0^1) \arrow[u] \arrow[r] & \cdots \arrow[r] & 0 \\
d' \uparrow & f \in \arrow[u, bend left] & {} \arrow[r] & (A_1, N_{-1}^0) \arrow[u] \arrow[r] & (A_1, N_{-1}^1) \arrow[u] \arrow[r] & \cdots \arrow[r] & 0 \\
 & & & {} \arrow[u] & {} \arrow[u] & & \\
 & & & \xrightarrow{\quad} d'' & & &
\end{tikzcd}$$

$q = -1$

$d' \nearrow$ $\quad p = 0$

$$\begin{tikzcd}
 & 0 & 0 & & \\
{} \arrow[r] & (A_0, N_1^0) \arrow[u] \arrow[r] & (A_0, N_0^1) \arrow[u] \arrow[r] & \cdots \arrow[r] & 0 \\
{} \arrow[r] & (A_1, N_0^0) \arrow[u] \arrow[r] & (A_1, N_{-1}^1) \arrow[u] \arrow[r] & \cdots \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & &
\end{tikzcd}$$

$\xrightarrow[d'']{}$

$d := d' + d''$

$$GN = \left(\cdots \longrightarrow (GN)^p \xrightarrow{d^p} (GN)^{p+1} \longrightarrow \cdots \right)$$

We can easily check that the assignment $N \longmapsto GN$ defines

an additive functor $G.\ C^{\uparrow}(A') \longrightarrow C^{\downarrow}(A)$.

$\square$

Step 5 Prove that G induces triangulated functors $K^{\uparrow}(A^!) \longrightarrow K^{\downarrow}(A)$ and $D^{\uparrow}(A^!) \longrightarrow D^{\downarrow}(A)$.

Step 6 Prove that $F : C^{\downarrow}(A) \longrightarrow C^{\uparrow}(A^!)$ is a left adjoint to

$$G : C^{\uparrow}(A^!) \longrightarrow C^{\downarrow}(A).$$

Let $M \in C^{\downarrow}(A)$ and $N \in C^{\uparrow}(A^!)$.

$$\underset{\substack{\rotatebox{90}{$\in$}\\ \tilde f}}{\mathrm{Hom}_{A^!}}\big(\overbrace{A^!\otimes M}^{\substack{FM\\ \shortparallel}}, N\big) \cong \mathrm{Hom}_k\big(M, \mathrm{Hom}_{A^!}(A^!, N)\big)$$

$$\cong \mathrm{Hom}_k(M, N)$$

$$\underset{f}{\rotatebox{90}{$\in$}}$$

$$\cong \mathrm{Hom}_k(A \underset{A}{\otimes} M, N)$$

$$\cong \mathrm{Hom}_A\big(M, \mathrm{Hom}_k(A, N)\big)$$

$$= \underset{\substack{\rotatebox{90}{$\in$}\\ \hat f}}{\mathrm{Hom}_A}\big(M, \underbrace{\bigoplus_{\ell\geq 0} \mathrm{Hom}_k(A_\ell, N)\big)}_{\substack{\shortparallel\\ GN}}$$

$$\tilde f \mapsfrom f \qquad f \leftrightarrow \hat f$$

☺ $M \in C^{\downarrow}(A)$

We can check that :

$$(1)\quad \tilde{f}\big(\underbrace{(FM)^p_q}_{\substack{\shortparallel \\ \bigoplus_{\substack{p=i+j \\ q=\ell-j}} A^!_\ell \otimes M^i_j}}\big) \subseteq N^p_q \quad \forall p,q \quad \Longleftrightarrow \quad \hat{f}(M^p_q) \subseteq \underset{\substack{\shortparallel \\ \bigoplus_{\substack{p=i+j \\ q=\ell-j}} \mathrm{Hom}_k(A_{-\ell}, N^i_j)}}{(GN)^p_q} \quad \forall p,q$$

(2) $\tilde{f}$ commutes the differentials on FM and N if and only if

$\hat{f}$ ________________________ M and GN.

$$\therefore \quad \mathrm{Hom}_{C^{\uparrow}(A^!)}(FM, N) \cong \mathrm{Hom}_{C^{\downarrow}(A)}(M, GN). \qquad \square$$

Step 7 Prove that the counit $\varepsilon_N : FGN \longrightarrow N$ is a quism for any $N \in C^{\uparrow}(A^!)$.

(1)

$$\begin{tikzcd}
\mathrm{Hom}_k(\overbrace{GN}^{\overset{C^{\downarrow}(A)}{\rotatebox{90}{$\in$}}}, N) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & \mathrm{Hom}_A(GN, \mathrm{Hom}_k(A,N)) \arrow[d, phantom, "\rotatebox{90}{$=$}"] \\
\mathrm{Hom}_{A^!}(A^! \otimes GN, N) \arrow[r, phantom, "\circlearrowleft"] & \mathrm{Hom}_A(GN, GN) \\
\mathrm{Hom}_{C^{\uparrow}(A^!)}(FGN, N) \arrow[u, hook] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathrm{Hom}_{C^{\downarrow}(A)}(GN, GN) \arrow[u, hook] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\varepsilon_N & \mathrm{Id}_{GN} \arrow[l, mapsto]
\end{tikzcd}$$

$a \in A^!$, $f \in GN$ $\quad \varepsilon_N(a \otimes f) = af(1_A)$.

(2)

$$\begin{tikzcd}
N \arrow[r, "\sim"] & k\otimes \mathrm{Hom}_k(k,N) \arrow[r, hook, "\mathrm{can}"] & A^! \otimes \Big(\overbrace{\bigoplus_{\ell\geq 0} \mathrm{Hom}_k(A_\ell, N)}^{\substack{\bar{g}_n \in \mathrm{Hom}_k(A,N) \\ \cup}}\Big) = FGN \\
N^i_j \arrow[u] \arrow[rr] & & k\otimes \mathrm{Hom}_k(k, N^i_j) \subset (FGN)^i_j \arrow[u]
\end{tikzcd}$$

$\theta : N \xrightarrow{\sim} k\otimes \mathrm{Hom}_k(k,N) \xhookrightarrow{\mathrm{can}} A^! \otimes \left(\bigoplus_{\ell\geq 0} \mathrm{Hom}_k(A_\ell, N)\right) = FGN$

$n \longmapsto 1\otimes g_n \longmapsto 1_A \otimes \bar{g}_n$

$\left(\begin{array}{l} g_n : k \longrightarrow N \\ \quad\;\; 1 \longmapsto n \end{array} \right)$

θ is a chain map in k-Gr.

$$\begin{tikzcd}
(FGN)^i_j \arrow[r, "d^i_j"] & (FGN)^{i+1}_j \\
N^i_j \arrow[u, "\theta^i_j"] \arrow[r, "\partial^i_{N_j}"'] \arrow[ur, phantom, "\circlearrowright"] & N^{i+1}_j \arrow[u, "\theta^{i+1}_j"']
\end{tikzcd}$$

$FGN \xrightarrow{\varepsilon_N} N \qquad \varepsilon_N(1\otimes \bar{g}_n) = \bar{g}_n(1_A) = g_n(1) = n$

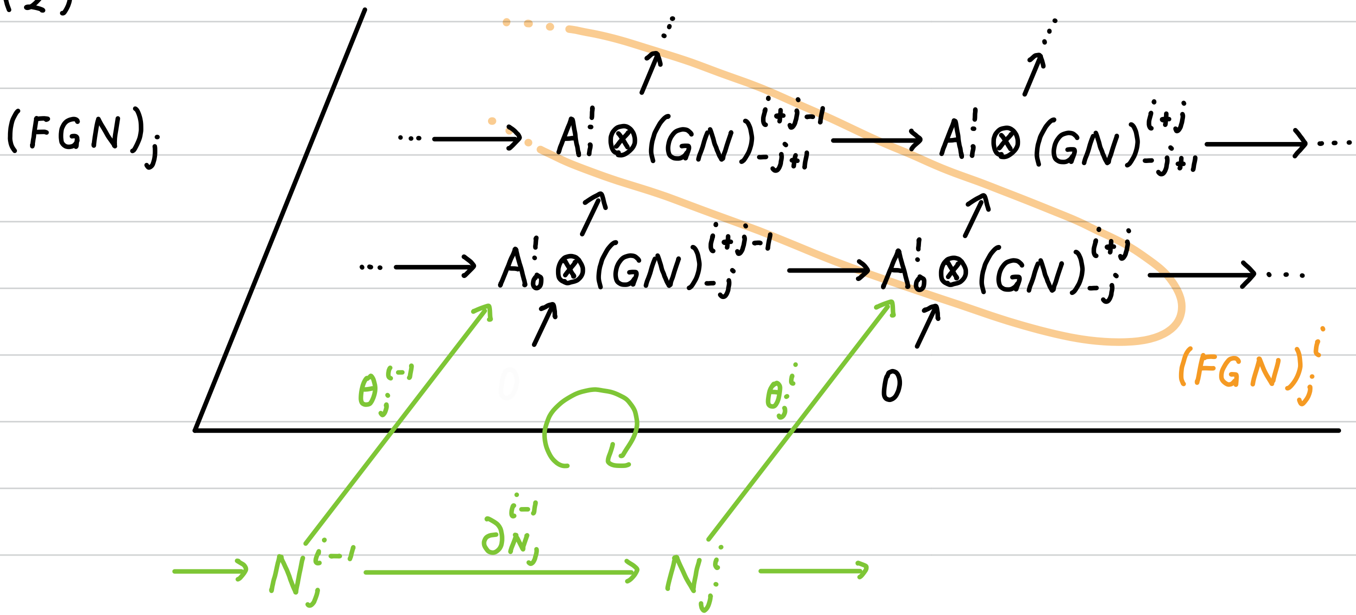
$N \xhookrightarrow{\theta} FGN \overset{\varepsilon_N}{\twoheadrightarrow} N \qquad \varepsilon_N \theta = Id_N$

θ is a quism iff $\underline{\varepsilon_N \text{ is a quism}}$. ← Want to show

θ is a quism iff θ_j is a quism for all $j \in \mathbb{Z}$

Enough to show

(2)



This is a morphism of double complexes whose totalization is θ_j.

By a standard argument of double complexes, θ_j is a quism if the sequence

$$0 \longrightarrow N_j^i \longrightarrow A_0^! \otimes (GN)^{i+j}_{-j} \longrightarrow A_1^! \otimes (GN)^{i+j}_{-j+1} \longrightarrow \cdots \qquad (i,j)$$

is exact for all $i, j \in \mathbb{Z}$.

(3) (i,j) is exact for all $i,j \in \mathbb{Z}$ iff

$$0 \longrightarrow \bigoplus_{p=i+j} N_j^i \longrightarrow A_0^! \otimes (GN)^p \longrightarrow A_1^! \otimes (GN)^p \longrightarrow \cdots \qquad \bigoplus_{p=i+j} (i,j)$$

exact for all $p \in \mathbb{Z}$.

$$\begin{tikzcd}
 & \overline{N}^p \arrow[d, equal] & & & \\
0 \arrow[r] & \bigoplus\limits_{p=i+j} N_j^i \arrow[r] \arrow[dddd, "\wr"] & A_0^! \otimes (GN)^p \arrow[r] \arrow[d, equal] & A_1^! \otimes (GN)^p \arrow[r] \arrow[dddd, "\wr"] & \cdots \\
 & & A_0^! \otimes \Big(\bigoplus\limits_{q} \Big(\bigoplus\limits_{\substack{p=i+j \\ q=\ell-j}} \mathrm{Hom}_k(A_{-\ell}, N_j^i) \Big) \Big) \arrow[d, equal] & & \\
 & {} \arrow[r, phantom, "\circlearrowright"] & A_0^! \otimes \Big(\bigoplus\limits_{\ell} \mathrm{Hom}_k(A_{-\ell}, \overline{N}^p) \Big) \arrow[d, "\wr"] \arrow[r, phantom, "\circlearrowright"] & {} & \\
 & & {} & & \\
0 \arrow[r] & \mathrm{Hom}_k(k, \overline{N}^p) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus\limits_{\ell} \mathrm{Hom}_k(A_{-\ell} \otimes \underbrace{{}^*(A_0^!)}_{\overset{\shortparallel}{k}}, \overline{N}^p) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus\limits_{\ell} \mathrm{Hom}_k(A_{-\ell} \otimes {}^*(A_1^!), \overline{N}^p) \arrow[d, hook] & \\
0 \arrow[r] & \mathrm{Hom}_k(k, \overline{N}^p) \arrow[r] & \mathrm{Hom}_k(A, \overline{N}^p) \arrow[r] & \mathrm{Hom}_k(A \otimes {}^*(A_1^!), \overline{N}^p) \arrow[r] & \cdots : \mathrm{ex.} \\
0 & k \arrow[l] & A \arrow[l] & A \otimes {}^*(A_1^!) \arrow[l] & \cdots : \mathrm{ex.} \arrow[l]
\end{tikzcd}$$

Koszul complex

$$\begin{tikzcd}
 & \displaystyle\sum_{-t\le \ell\le 0} h_\ell \arrow[r, mapsto] & 0 \\
\displaystyle\bigoplus_{\ell} \mathrm{Hom}_k(A_{-\ell}\otimes^*(A^{!}_{i-1}), \overline{N}^{P}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & \displaystyle\bigoplus_{\ell} \mathrm{Hom}_k(A_{-\ell}\otimes^*(A^{!}_{i}), \overline{N}^{P}) \arrow[r] \arrow[d, hook] \arrow[u, phantom, sloped, "\in"] \arrow[dr, phantom, "\circlearrowright"] & \displaystyle\bigoplus_{\ell} \mathrm{Hom}_k(A_{-\ell}\otimes^*(A^{!}_{i+1}), \overline{N}) \arrow[d, hook] \\
\mathrm{Hom}_k(A\otimes^*(A^{!}_{i-1}), \overline{N}^{P}) \arrow[r] & \mathrm{Hom}_k(A\otimes^*(A^{!}_{i}), \overline{N}^{P}) \arrow[r] & \mathrm{Hom}_k(A\otimes^*(A^{!}_{i+1}), \overline{N}^{P}) : \text{exact}
\end{tikzcd}$$

$$\begin{tikzcd}
{}^{\exists}g \arrow[r, mapsto] & \displaystyle\sum_{-t\le \ell\le 0} h_\ell & \\
\displaystyle\bigoplus_{-\ell+i-1\le m} \mathrm{Hom}_k(A_{-\ell}\otimes^*(A^{!}_{i-1}), \overline{N}^{P}) \arrow[r] \arrow[d, "\wr"] \arrow[u, phantom, sloped, "\in"] \arrow[dr, phantom, "\circlearrowright"] & \displaystyle\bigoplus_{-\ell+i\le m} \mathrm{Hom}_k(A_{-\ell}\otimes^*(A^{!}_{i}), \overline{N}^{P}) \arrow[d, "\wr"] \arrow[u, phantom, sloped, "\in"] & {}^{\exists}m\ge 0 \\
\mathrm{Hom}_k((A\otimes^*(A^{!}_{i-1}))_{\le m}, \overline{N}^{P}) \arrow[r] & \mathrm{Hom}_k((A\otimes^*(A^{!}_{i}))_{\le m}, \overline{N}^{P}) \arrow[r] & \mathrm{Hom}_k((A\otimes^*(A^{!}_{i+1}))_{\le m}, \overline{N}^{P}) : \text{exact}
\end{tikzcd}$$

$\square$

Step8 Prove that the unit $M \longrightarrow GFM$ is a quism for any $M \in C^{\downarrow}(A)$

By Steps 1-8, we have :

Thm ([BGS])

$$\begin{tikzcd}
F : D^{\downarrow}(A) \arrow[r, shift left=1.5ex, "\sim"] \arrow[r, phantom, "\perp"] & D^{\uparrow}(A^{!}) : G \arrow[l, shift left=1.5ex, "\sim"]
\end{tikzcd}$$

Koszul duality functor

Rem (i) $F(M(n)) = (FM)(-n)[n] \quad \forall n \in \mathbb{Z}$

(ii) $Fk \cong A^{!}$, $F(A^{\circledast}) \simeq k = A^{!}_{0}$

Lem

(i) $A \in k\text{-mod} \Rightarrow D^b(A\text{-gr}) \xrightarrow{\sim} D^{\downarrow}_e(A) := \{ M \in D^{\downarrow}(A) \mid \bigoplus_i H^i(M) \in A\text{-mod} \}$

(ii) $A^! : \text{left noeth.} \Rightarrow D^b(A^!\text{-gr}) \xrightarrow{\sim} D^{\uparrow}_e(A^!) := \{ M \in D^{\uparrow}(A^!) \mid \bigoplus_i H^i(M) \in A^!\text{-mod} \}$

Cor

$A \in k\text{-mod}$, $A \in k^{op}\text{-mod}$, $A^! : \text{left noeth.}$

$$\begin{tikzcd}
D^{\downarrow}(A) \arrow[rr, "\overset{F}{\sim}"] & & D^{\uparrow}(A^!) \\
\cup & & \cup \\
D^{\downarrow}_e(A) \arrow[r, phantom, "\circlearrowleft"] & {} & D^{\uparrow}(A^!) \\
D^b(A\text{-gr}) \arrow[u, "\wr"] \arrow[rr, "\sim"] & & D^b(A^!\text{-gr}) \arrow[u, "\wr"]
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
D^{\downarrow}(A) \arrow[rr, "\overset{F}{\sim}"] & & D^{\uparrow}(A^!) \\
\cup & \circlearrowright & \cup \\
D^{\downarrow}_e(A) \arrow[rr, hook] & & D^{\uparrow}_e(A^!)
\end{tikzcd}$$

Since $F(k) \cong A^!$, it suffices to show:

<u>Claim</u> Every $M \in A^!\text{-gr}$ has finite projective dimension.

$\left(\text{ETS: } \exists d > 0 \text{ s.t. } \operatorname{ext}^i_{A^!}(k, k(j)) = 0 \ \forall i > d \ \forall j \in \mathbb{Z} \right)$

Proof of Claim.

$$\mathrm{Hom}_{D^b(A\text{-gr})}(A^{\circledast}, A^{\circledast}(-j)[i+j]) \overset{F}{\xrightarrow{\sim}} \mathrm{Hom}_{D^b(A^!\text{-gr})}(k, k(j)[i])$$

$$\|$$

$$0 \quad \text{if } i+j \neq 0$$

$$i+j=0 \Rightarrow \mathrm{Hom}_{D^b(A\text{-gr})}(A^{\circledast}, A^{\circledast}(i)) \overset{F}{\xrightarrow{\sim}} \mathrm{Hom}_{D^b(A^!\text{-gr})}(k, k(-i)[i])$$

$$\|$$

$$\mathrm{hom}_A(A^{\circledast}, A^{\circledast}(i))$$

$$\|$$

$$\mathrm{hom}_k(A^{\circledast}(-i), k)$$

$$\|$$

$$((A^*)_i)^*$$

$\therefore \exists d>0$ s.t.

$$\mathrm{Hom}_{D^b(A^!\text{-gr})}(k, k(j)[i]) = 0$$

$\forall i>d$, $\forall j \in \mathbb{Z}$. □

t-structures

$D^{\uparrow}(A) \supset D^{\uparrow}(A)^{\leq 0, g} \ni X \overset{\text{def}}{\iff} X \in D^{\uparrow}(A),\ X \cong^{\exists} P \in D^{\uparrow}(A)$ s.t.

$\left(\text{resp. } D^{\uparrow}(A)^{\geq 0, g} \ni X \right)$ P^{i} : projective , $P^{i} = AP^{i}_{\leq -i}$, $\forall i \in \mathbb{Z}$

$\left(\text{resp. } P^{i} = AP^{i}_{\geq -i} \right)$

Prop ([BGS])

$$\left(D^{\downarrow}(A)^{\leq 0},\ D^{\downarrow}(A)^{\geq 0} \right) \xrightarrow[\sim]{F} \left(D^{\uparrow}(A^{!})^{\leq 0, g},\ D^{\uparrow}(A^{!})^{\geq 0, g} \right)$$

the standard t-structure a non-standard t-structure