Koszulity for preprojective algebras and zigzag algebras

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Koszulity for preprojective algebras and zigzag algebras

Aim

Q : finite quiver.

- Q : non-Dynkin $\Rightarrow$ PPA & ZZA Koszul
- Q : Dynkin $\Rightarrow$ almost "

preprojective algebra (=: PPA) $\overset{\text{quadratic dual}}{\longleftrightarrow}$ zigzag algebra (=: ZZA)

- PPA has several characterizations from defferent viewpoints and gives many interests to defferent areas.

(I) PPA & ZZA (they are quad. dual)

(II) Introduce 3 characterizations of PPA and induce Koszulity from them.

① Skew-group algebra. (Q : ext. Dynkin, $\Bbbk = \overline{\Bbbk}$ $\mathrm{ch}\Bbbk = 0$)

② Calabi-Yau completion. (Q : acyclic)

③ Rep theory of Q (Q : acyclic)

(III) Almost Koszulity of PPA of Dynkin type.

(I). PPA & ZZA.

Def $Q = (I, E)$ quiver (oriented connected finite graph)

for simplicity

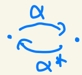
I : vertex, E : edge

$h \xleftarrow{\alpha} t$ head tail

$\rightsquigarrow$ We define the path algebra of Q over a field $\Bbbk$ as the tensor algebra.

$$\Bbbk Q := T_S(E)$$

- $S = \bigoplus_{i \in I} \Bbbk e_i$ ← path of length 0.
- For $\alpha \in E$, $\alpha e_j = \delta_{t(\alpha) j}\, \alpha$, $e_j \alpha = \delta_{h(\alpha) j}\, \alpha$

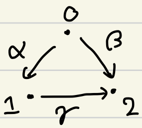
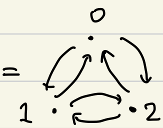


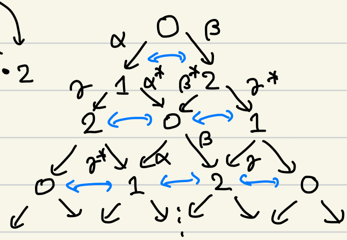
We take the double $\overline{Q}$ of Q as $\begin{cases} \overline{I} = I \\ \overline{E} = E \amalg E^* \end{cases}$ opposite

and define the following quadratic algebra Π_Q (PPA)

<u>Def</u> (Gelfand - Ponomarev)

$$\Pi = \Pi_Q := \Bbbk\overline{Q} \Big/ \Big\langle \sum_{\alpha \in E_Q} \alpha\alpha^* - \alpha^*\alpha \Big\rangle$$

(e.g.) $Q =$  $\overline{Q} =$ 

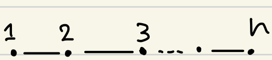
Π has paths like $\rightarrow$ 

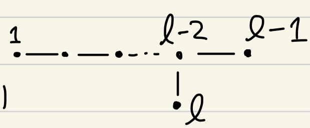
<u>Rep</u>
- Π does not depend on the choice of orientation of Q (up to isomorphisms)
- Π contains $\Bbbk Q$ as a subalgebra for any orientation.

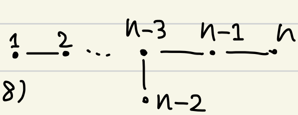
- ADE classification often gives characterizations of nice homological classes of quiver algebras

<u>Def</u> Q: Dynkin type

$:\Leftrightarrow$ underlying graph Q° of Q is one of

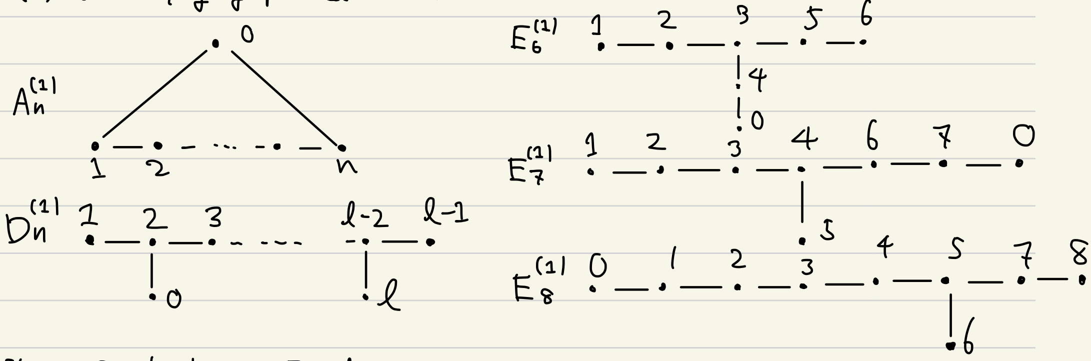
A_n 

D_n $(n \geq 4)$ 

E_n $(n = 6, 7, 8)$ 

Def Q : extended Dynkin type.

$:\Longleftrightarrow$ underlying graph Q° of Q is one of



e.g.) [Baer-Geigle-Lenzing] $\dim_{\Bbbk} \Pi_Q < \infty \Longleftrightarrow Q$: Dynkin,

Π_Q : Noetherian of finite Gelfand-Kirillov dimension (cf. [MR])

$\Longleftrightarrow Q$: Dynkin or ext Dynkin.

[Bocklandt] $\Bbbk = \overline{\Bbbk}$, Λ : quiver alg. w/ suitable grading

$D^b(\Lambda\text{-mod}_{fd})$: 2-Calabi-Yau $\Longleftrightarrow \Lambda$: PPA (Q : non-Dynkin)

[Białkowski-Erdmann-Skowroński] Λ : self-inj. fin-dim'l alg

Then, $\Omega^3_\Lambda(S) \simeq \nu_\Lambda(S)$ ($\forall S$: non proj. simple Λ-mod)

$\Longleftrightarrow \Lambda$: "deformed" PPA (Q : "generalized" Dynkin)

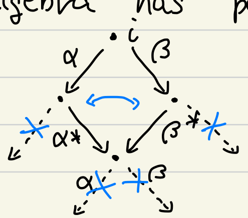
$\Lambda^* \otimes_\Lambda -$: $\mathrm{proj}\,\Lambda \xrightarrow{1:1} \mathrm{inj}\,\Lambda$

Def

Q : acyclic quiver.

$$Z_Q := \Bbbk\overline{Q} \Big/ \left\langle \begin{array}{ll} \alpha\beta & (\beta \neq \alpha^*, (\alpha^*)^* = \alpha) \\ \varepsilon(\alpha)\alpha^*\alpha - \varepsilon(\beta)\beta^*\beta & (t(\alpha) = t(\beta)) \end{array} \right\rangle \qquad \varepsilon(\alpha) = \begin{cases} 1 & (\alpha \in Q) \\ -1 & (\alpha \in Q^*) \end{cases}$$

This algebra has paths like



Z_Q is a $2(|I| + |E|)$ dimensional $\Bbbk$-algebra.

$\rightsquigarrow$ The quadratic dual of Z_Q is

$$\Bbbk\overline{Q} \Big/ \left\langle \sum_{\alpha \in E_Q} \alpha\alpha^* - \alpha^*\alpha \right\rangle \quad \text{(preprojective algebras)}$$

Rem
- The algebra Z_Q is called the zigzag algebra.
- The algebra Z_Q does not depend on our choice of orientation up to isom.
- The algebra Z_Q is isomorphic to the trivial extension alg. of $\mathbb{k}Q/(\mathrm{rad}\,\mathbb{k}Q)^2$.

- $\exists\, \chi : Z_Q \longrightarrow \mathbb{k}$

$$\begin{aligned} \varepsilon(\alpha)\alpha^*\alpha &\longmapsto 1 \\ \alpha,\ e_i &\longmapsto 0. \end{aligned} \quad \rightsquigarrow \quad \exists \langle\ ,\ \rangle : Z_Q \times Z_Q \longrightarrow \mathbb{k} \ \text{(nondegenerate)}$$

$\therefore Z_Q$ is a Frobenius algebra.

Rem In tomorrow lectures, we will consider Koszul dualities about Frobenius Koszul algebras & AS-regular alg's in more general settings.

(II). After H. Nakajima proposed some question about Koszulities of PPAs in ICRA (1994) and [Martínez-Villa] (bipartite case), many people proved "Koszulities" of PPAs from defferent viewpoints.

Rem In general, PPA for any finite non-Dynkin quiver (not necessarily acyclic) is Koszul. ([Etingof-Eu]).
But we will NOT discuss their argument.

- <u>Golod-Shafarevich inequalities.</u>

Λ : $\mathbb{Z}_+$-graded alg. w/ $\Lambda_0 : S = \mathbb{k}^{|I|}$ "n"

For $M \in S,S$ bimod, $[M] := (\dim_{\mathbb{k}} e_i M e_j)_{ij} \in \mathrm{Mat}(n, \mathbb{Z})$

$\underset{i \in \mathbb{Z}}{\oplus} M_i$ //

For M: graded S-bimod, $P(M,t) := \sum_i [M_i] t^i$ (matrix Hilb. series)

Lem $\Lambda : T_S(V)/(R)$ quad. algebra

deg 1 / deg 2

(i). $\dfrac{1}{1-[V]t+[R]t^2} \geq 0 \Rightarrow P(\Lambda,t) \geq \dfrac{1}{1-[V]t+[R]t^2}$

(ii). $P(\Lambda,t) = \dfrac{1}{1-[V]t+[R]t^2} \Rightarrow \Lambda$: Koszul.

(:) (*)

$$0 \longrightarrow {}^{\exists}K \longrightarrow \Lambda \otimes_S R \longrightarrow \Lambda \otimes_S V \longrightarrow \Lambda \longrightarrow S \longrightarrow 0$$

(exact Koszul cpx)
(see later)

$$P(\Lambda,t)\cdot 1 - P(\Lambda,t)\cdot[V]t + P(\Lambda,t)\cdot[R]t^2 - 1 = P(K,t) \geq 0.$$

(ii) //

If $P(\Lambda,t) = \dfrac{1}{1-[V]t+[R]t^2} \Rightarrow K=0$.

$(*) \xrightarrow{S\otimes_\Lambda -} \quad 0 \to R \xrightarrow{0} V \xrightarrow{0} S \to 0$

$\therefore \operatorname{Tor}_i^\Lambda(S,S)$ is concentrated in degree i □.

① Skew-group alg. ([Reiten-Van-den-Bergh, Crawley-Boevey-Holland, ...])

Overview

$\Bbbk = \overline{\Bbbk}$, $\operatorname{ch}\Bbbk = 0$. $SL_2(\Bbbk) \overset{\text{finite sub}}{\supset} \Gamma \overset{\text{McKay corr.}}{\longleftrightarrow} \{\text{ext. Dynkin diagram}\}$

$\Bbbk[X,Y]^\Gamma \overset{\text{non comm. deformation}}{\rightsquigarrow} \Bbbk[X,Y] \rtimes \Gamma =: \Lambda$

quotient singularity (In some sense smooth.

$$\Bbbk[X,Y]^\Gamma \simeq e_0 \Lambda e_0 \quad (\simeq \Bbbk[\operatorname{Rep}(\Pi_Q, \delta)]^{GL(\delta)})$$

${}^{\exists}$dim. vector.

idempotent corresponding to trivial rep

corresponding ext. Dynkin.

Fact [Reiten-Van-den-Bergh]

$\Bbbk = \overline{\Bbbk}$, $\mathrm{ch}\,\Bbbk = 0$.

McKay corr.

Q : extended Dynkin. $\exists!$ Γ : finite subgrp of $SL_2(\Bbbk)$

V : vector rep. of Γ $(a\otimes g)(b\otimes h) = ag(b)\otimes gh$

$$\begin{tikzcd} \bigwedge^2 V^* \rtimes \Gamma \arrow[r, leftrightarrow, "\text{quad.}"] \arrow[d, "\text{Morita}"] & S^2V \rtimes \Gamma \simeq \Bbbk[\overset{1}{x}, \overset{1}{y}] \rtimes \overset{0}{\Gamma} \; (=: \Lambda) \arrow[d, "\text{Morita.}"] \\ Z_Q \arrow[r, leftrightarrow, "\text{quad.}"] & \Pi_Q \end{tikzcd}$$

$\leftarrow$ equipped w/ path grading.

- Koszulity of Π_Q is induced from that of $\Bbbk[x,y]$ ([Etingof-Eu])

sketch) $\Bbbk\Gamma \otimes - \curvearrowright$ (Koszul cpx of $\Bbbk[x,y]$)

$A_Q := (a_{ij})$
$a_{ij} := \#\{\text{edges connecting } i \,\&\, j\}$

$$K^\bullet : 0 \longrightarrow \Lambda(-2) \longrightarrow \Lambda \otimes_{\Bbbk\Gamma} (\Lambda)_1 \longrightarrow \Lambda \longrightarrow \Bbbk\Gamma \to 0$$

Morita $\downarrow$

$$0 \longrightarrow \Pi_Q(-2) \longrightarrow \Pi_Q \otimes_S (\Pi_Q)_1 \longrightarrow \Pi_Q \longrightarrow S \to 0$$

$$\rightsquigarrow 1 = P(\Bbbk^n, t) = P(\Pi_Q, t) - P(\Pi_Q, t) A_Q t + P(\Pi_Q, t) t^2$$

$$\Leftrightarrow P(\Pi_Q, t) = \frac{1}{1 - A_Q t + t^2} \underset{\text{GS-ineq.}}{\rightsquigarrow} \Pi_Q : \text{Koszul}. \qquad \square$$

② Graded Calabi-Yau completion (Minamoto)

- It is known that Π_Q (Q: non-Dynkin) is a 2-Calabi-Yau algebra (This is a special case of Artin-Schelter regular algs)

($D^b(\Lambda\text{-mod}_{fd})$ has $[2]$ as the Serre functor
$\mathrm{Hom}_{D^b}(M,N)^* \simeq \mathrm{Hom}_{D^b}(N, M[2])$)

$\rightsquigarrow$ PPA can be thought as a procedure to obtain CY alg from an algebra.

If we consider path grading on ΠQ, then ΠQ (Q: Non-Dynkin) is a $(2,2)$-Calabi-Yau algebra.

($\mathrm{Hom}_{D^b}(M,N)^* \simeq \mathrm{Hom}_{D^b}(N, M[2](2))$
(This can be read from bimodule resl (cf. [Geiss-Leclerc-Schröer]))

- PPA has characterizations as "non-commutative analogue" of the cotangent bdl of an affine curve [Crawley-Boevey et.al]
"$\Pi Q \simeq T_{\Bbbk Q}(H^1 \Omega)$ (Ω: inverse dualizing cpx)"

$\rightsquigarrow$ There is a natural framwork to obtain a CY dga from a dga (defferential graded algebra).

<u>Fact</u> ([Keller] (cf. Ikeda-Qiu))

defferential bigraded algebra

A: (homologically smth) dbga $\rightsquigarrow$ $^\exists\, \Pi_{(m,n)}$(bimodule) (m,n)CY-dbga.
$(d: A_n^m \to A_n^{m-1})$

$\Pi_{(m,n)}(A) = T_A(\omega)$ ($\omega :=$ cofibrant resl. of $\Omega_A[m-1](-n)$ ($\Omega_A = \mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)$))
$(A^e = A \otimes_{\Bbbk} A^{op})$

homological Adams

We take Q as a non-Dynkin acyclic quiver. then it is known that $\Pi_{(m,n)}(\Bbbk Q) \simeq T_S(\underbrace{\Bbbk E(-1) \oplus \Bbbk E^*[m-2](-n+1) \oplus S[m-1](n)}_{(*)})$

$\simeq \Bbbk \widetilde{Q}$

$$\left(\begin{array}{l}
\widetilde{Q}_0 = Q_0 \\
\widetilde{Q}_1 = \underset{(0,1)}{Q_1} \amalg \underset{(-m+2,\, n-1)}{Q_1^*} \amalg \underset{(-m+1,\, n)}{\{ \overset{t_i}{\circlearrowleft} \mid i \in Q_0 \}}. \\
\\
d \circlearrowright k\widetilde{Q} \\
\quad \text{by} \quad d(\alpha) = 0 = d(\alpha^*) \quad (\alpha \in Q_1) \\
\qquad\quad d(t_i) = e_i \left(\sum_{\alpha \in Q_1} [\alpha, \alpha^*] \right) e_i
\end{array}\right.$$

<u>Fact - Def.</u>

- $M = (\bigoplus_{n \in \mathbb{Z}} M^n, d_M)$: cpx of graded S-bimodules.

(1) M : suitably bounded $:\Leftrightarrow$ $^\forall i \in \mathbb{Z}$ $\dim_{\Bbbk} \bigoplus_{n \in \mathbb{Z}} M_i^n < \infty$.
(2). $^\exists i_0 \in \mathbb{Z}$, $M_i^\bullet = 0$ $(^\forall i < i_0)$

$P(M,t) := \sum_n (-1)^n P(M^n, t) = \sum_{i,n} (-1)^n [M_i^n] t^i$,

$$\begin{cases} \cdot\ P(M[n](m),t) = (-1)^n t^m \cdot P(M,t) \\ \cdot\ M, N : \text{quasi-isom} \Rightarrow P(M,t) = P(N,t) \\ \cdot\ P(M \otimes_S N, t) = P(M,t)\cdot P(N,t) \end{cases}$$

<u>Fact</u> Q : acyclic $\Longrightarrow$ $\Pi_Q \underset{\text{qis.}}{\simeq} \Pi_{(2,2)}$

$\rightsquigarrow P((*),t) = [E]t + [E^*]t - [S]t^2 = A_Q t - t^2$

$\therefore P(\Pi_Q, t) = P(\Pi_{(2,2)}, t)$

$= P(T_S((*)), t) = \sum_{n\geq 0} P((*),t)^n = \dfrac{1}{1-P((*),t)} = \dfrac{1}{1-A_Q t + t^2}$ $\rightsquigarrow$ Π_Q : Koszul

③ Rep theory of Q ([Brenner-Buttler-King])

- $\Pi_Q / {}^{\exists}I$ (ideal) $\simeq \Bbbk Q$, $\Bbbk Q \overset{\text{sub alg.}}{\hookrightarrow} \Pi_Q$

$\rightsquigarrow$ Π_Q "contains" rep theory of Q for any orientation.

- We assume Q : acyclic quiver.
  - Π_Q is the directsum of the preproj. modules of $\Bbbk Q$ $(=: \Lambda)$

<u>Fact</u> ([Baer-Geigle-Lenzing, Crawley-Boevey, Ringel])

We call it BGL description

- $\Pi \simeq \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathrm{Hom}_\Lambda(\Lambda, \tau_\Lambda^{-n}\Lambda) \simeq \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \tau_\Lambda^{-n}\Lambda$ (as left Λ-mod)

($\tau_\Lambda^{-} := \mathrm{Ext}^1_\Lambda(\Lambda^*, -)$ (cf. [ASS])

$\rightsquigarrow$ Π : fin. dim'l $\Longleftrightarrow$ Q : finite Dynkin

- In general, Q : fin. Dynkin $\Rightarrow$ Π : self-inj. ($\Pi \simeq \Pi^*$ as Π^{op}-mod)

We consider projective resol. of simple Π-modules.

<u>Prop</u> ${}^{\forall} i \in I$,

$$e_i\Pi \xrightarrow[\varphi]{} \bigoplus_{i\sim j} e_j\Pi \longrightarrow e_i\Pi \longrightarrow S_i \longrightarrow 0. \quad \text{(exact)}$$

(i). Q : Dynkin $\Rightarrow$ $\mathrm{Ker}\,\varphi \simeq S_{\sigma(i)}$ ($(e_i\Pi)^* \simeq \Pi e_{\sigma(i)}$)

(ii) Q : non-Dynkin $\Rightarrow$ 〃 $= 0$.

<u>Pf</u> Choose orientation as $\searrow \underset{i}{\cdot} \swarrow$ sink. (Λe_i : simple proj)

$$0 \to \Lambda e_i \to \bigoplus_{j\to i} \Lambda e_j \to \tau_\Lambda^{-1}(\Lambda e_i) \to 0.$$

$\downarrow \mathrm{Hom}_\Lambda(-,\Pi)$

$$0 \to \underbrace{\mathrm{Hom}_\Lambda(\tau^{-1}(\Lambda e_i),\Pi)}^{(*)} \to \bigoplus \underbrace{\mathrm{Hom}_\Lambda(\Lambda e_j,\Pi)}_{e_j\Pi} \to \underbrace{\mathrm{Hom}(\Lambda e_i,\Pi)}_{e_i\Pi}$$

$$\to \underbrace{\mathrm{Ext}^1_\Lambda(\ -\text{"}-\ )}_{1\text{-dim'l}} \to 0. \quad \simeq S_i$$

Q: non Dynkin $\Rightarrow \tau_\Lambda \Pi \simeq \Pi \quad \therefore (*) \simeq e_i\Pi$

Q: Dynkin

$\tau(\text{proj}) = 0$

$$\Rightarrow \quad 0 \to D\Lambda \to \Pi \xrightarrow[\text{(split)}]{} \tau_\Lambda\Pi \to 0$$

$\mathrm{Hom}_\Lambda(\Lambda e_i, -)$ ($\Lambda e_i \simeq S_i$)

$$0 \to \mathrm{Hom}_\Lambda(\Lambda e_i, D\Lambda) \to \mathrm{Hom}(\Lambda e_i,\Pi) \to \mathrm{Hom}(\Lambda e_i, \tau\Pi) \to 0$$

$\mathrm{Hom}_\Lambda(\Lambda e_i, D\Lambda) \simeq \mathrm{Soc}(i) \hookrightarrow \mathrm{Hom}(\Lambda e_i,\Pi) \simeq e_i\Pi$, $\quad \mathrm{Hom}(\Lambda e_i, \tau\Pi) \simeq \mathrm{Hom}(\tau^{-}(\Lambda e_i),\Pi)$

<u>Lem</u>

$A = T_S(V)/R$: R: minimal relation.

$\Rightarrow A\otimes R \xrightarrow{\iota} A\otimes V \xrightarrow{m} A \to 0$ gives. first 3-term of minimal proj. resl of S. (graded cpx) $(\clubsuit)$ in A-Gr

ι induced by $A\otimes R \hookrightarrow A\otimes V \otimes_S V \to A\otimes_S V$

Pf) We consider degreewise :

- $(\clubsuit)$ has homology A_0 in deg. 0.
- $0 \to A_0\otimes V \xrightarrow{m} A_1 \to 0$

- We prove $A_{n-2}\otimes R \to A_{n-1}\otimes V \xrightarrow{m} A_n \to 0.$ (exact ; $n\geq 2$)

$R_n := (R)_n$

$$\begin{tikzcd}
0 \arrow[r] & R_{n-2}\otimes R \arrow[r] \arrow[d, "\iota_R"] & V^{n-2}\otimes R \arrow[r, "p"] \arrow[d, "\iota_V"] & A_{n-2}\otimes R \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & R_{n-1}\otimes V \arrow[r, "s"] \arrow[d, "m_R"] & V^{n-1}\otimes V \arrow[r, "q"] \arrow[d, "m_V"'] \arrow[d, "S"'] & A_{n-1}\otimes V \arrow[r] \arrow[d, "m"] & 0 \\
0 \arrow[r] & R_n \arrow[r] \arrow[ur, bend right] & V^n \arrow[r, "r"] & A_n \arrow[r] & 0
\end{tikzcd}$$

①

$q^{-1}(\mathrm{Ker}(m)) = R_n \subset V^{n-1}\otimes V$

$\rightsquigarrow \quad \mathrm{Ker}(m) = q(R_n) = q(\mathrm{Im}\, S + \mathrm{Im}\, \iota_V)$

$= q(\mathrm{Im}\,\iota_V) = \mathrm{Im}(\iota \circ p) = \mathrm{Im}(\iota)$ $\square$

Cor Q : non-Dynkin $\Longrightarrow$ Π_Q : Koszul.

Rem In rep. theory of fin. dim'l algebras, it is known that a fin. dim'l alg w/ gl.dim ≤ 2 is always quasi-hereditary. ([Dlab-Ringel])

Rem In ② & ③, we assumed "non-Dynkin" properties on Q. Some meanings of them in the contexts of CY completions and BGL descriptions of preproj. alg's can be easily understood in terms of $D^b(\Bbbk Q\text{-mod})$:

② $\quad Q$: acyclic quiver

$\Pi_2(Q)$: 2 CY completion of $\Bbbk Q$.

definition of CY completion & ([BGL, CB, Ringel])

$\Pi(Q) \underset{\text{qis}}{\simeq} \Pi_2(Q) \quad \Longleftrightarrow \quad \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathrm{Hom}_{D^b(\Bbbk Q)}(\Bbbk Q, \tau^{-n}\Bbbk Q)$ (τ : derived AR trans)

concentrate in deg 0 in $D^b(\Bbbk Q\text{-mod})$

$\Longleftrightarrow$ Q : non-Dynkin.

$\tau \circlearrowright K_0(\Bbbk Q\text{-mod})$ is written by Coxeter trans.

Q : Dynkin $\Longrightarrow$ $\tau^{-h} \simeq [2]$ in $D^b(\Bbbk Q\text{-mod})$

(h : Coxeter # of Q)

③ We define a grading on Π by $\begin{cases} \deg \alpha = 0 \\ \deg \alpha^* = 1. \end{cases}$ $(\alpha \in Q)$

degree k part of $\left(e_i \Pi \xrightarrow[\varphi]{} \bigoplus_{i \sim j} e_j \Pi \longrightarrow e_i \Pi \longrightarrow S_i \longrightarrow 0. \right)$

$\rightsquigarrow$ BGL description $\tau^{-k}(e_i \Bbbk Q) \longrightarrow \tau^{-k}\left(\bigoplus_{i \sim j} e_j \Bbbk Q\right) \longrightarrow \tau^{-k-1}(e_i \Bbbk Q) \longrightarrow \cdots$

AR-triangle in $D^b((\Bbbk Q)^{op}\text{-mod})$

$\rightsquigarrow$ Q : non Dynkin $\Leftrightarrow$ φ : injective.

(by recalling "Q : Dynkin $\Rightarrow$ $\tau^{-h} \simeq [2]$ in $D^b((\Bbbk Q)^{op}\text{-mod})$")

(III). Almost Koszulity of Dynkin type.

PPAs of Dynkin type are not Koszul. However, they are almost Koszul in the following sense:

<u>Def</u> ([Brenner-Butler-King])

A : $\mathbb{Z}_{\geq 0}$-graded ring. w/ $A_0 =: S$ (semi-simple Artin)

A : left (p, q)-Koszul.

$:\Leftrightarrow$ (1). $n > p \Rightarrow A_n = 0$.

(2). $\exists$ graded cpx $P^\bullet : 0 \to P^q \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \to 0$.

s.t. · P^i : proj. generated by P^i_i (deg i component)

· non-zero homology is $\begin{cases} A_0 & (\deg 0) \\ A_p \otimes P^q_q & (\deg p+q) \end{cases}$

<u>Rem</u> $p \geq 2 \Rightarrow P^{q+1}$ ($q+2$-th resl of S) is generated in

degree $q+p. > q+1.$

$0 \searrow A_{p-1}\otimes P_q^q \searrow A_p \otimes P_{q-1}^{q-1}$ $p+q-1$ exact

$0 \searrow A_p \otimes P_q^q$ $p+q.$

- - - - - exact

$A_0 \otimes P_1^1 \searrow A_1 \otimes P_0^0 \searrow 0$ 1 exact

$A_0 \otimes P_0^0 \searrow 0$ 0

e.g.) V : S-S-bimodule

$$A = T_S(V) / V^{\otimes p+1} \quad (r \in \mathbb{Z}_{>0}) \quad \text{is} \quad (p,1)\text{-Koszul.}$$

$$(0 \longrightarrow A_p \otimes V \longrightarrow A \otimes V \longrightarrow A \longrightarrow A_0 \longrightarrow 0)$$

Coxeter #

<u>Thm</u> PPAs of Dynkin type are $(h-2, 2)$-Koszul.
(and so ZZA of Dynkin type are $(2, h-2)$-Koszul

$$\left(0 \longrightarrow \underset{h}{S_{\nu(i)}} \longrightarrow \underset{2}{P_i} \longrightarrow \underset{1}{\bigoplus P_j} \longrightarrow \underset{0}{P_i} \longrightarrow \underset{0}{S_i} \longrightarrow 0 \right)$$

This h comes from the degree of the socle of P_i
understood by AR-theory of $\Bbbk Q$ (see [BBK])