Part II BGG correspondense (Part I は後)

Thm (Berstein - Gelfand - Gelfand, 1978)
Let $n \geq 1$ be a natural number.
$V := \mathbb{k}^{n+1}$, $\mathbb{P}^n = \mathbb{P}(V) = \mathrm{proj}(\mathrm{Sym}\, V)$.
Then there exists the following equivalence of triangulated categories

$$D^b(\mathrm{coh}\, \mathbb{P}^n) \simeq \underline{\Lambda\ \mathrm{grmod}}$$

where $\Lambda := \dot{\wedge} V^*$ the exterior alg of the dual space $V^* = \mathrm{Hom}_{\mathbb{k}}(V, \mathbb{k})$ and $\underline{\mathrm{grmod}}$ denotes the "stable category".

AIM

Give a generalization and an outline of a proof.

PLAN
§1 Noncommutative projective scheme
§2 singular derived categories and stable categories
§3 Proof.

Setup $A = \bigoplus_{n \geq 0} A_n$, • graded Noetherian on both sides.
• $\dim_{\Bbbk} A_n < \infty$ ($\forall n$)
• A_0 is semi-simple

---

§1

Let $A\,\mathrm{fd} \subset A\,\mathrm{grmod}$ denote the full sub cat of finite $\Bbbk$-dimensional graded A-modules.

• $A\,\mathrm{fd}$ is a Serre sub category of $A\,\mathrm{grmod}$.

i.e. closed under taking
• sub object.
• quotient object.
• extensions.

Def (Artin-Zhang)

$$A\,\mathrm{qgr} := \frac{A\,\mathrm{grmod}}{A\,\mathrm{fd}}$$

"non commutative projective scheme associated to A"

• The quotient functor $\pi : A\,\mathrm{grmod} \longrightarrow A\,\mathrm{qgr}$ is exact.

- The degree shift functor $(1) \circlearrowright A$ grmod descents to $(1) \circlearrowright A$ qgr.

Rmk strictly speaking, the triple $(A\,\mathrm{qgr}, A, (1))$ is noncommutative proj. scheme of A. ┘

◎ If $\dim_{\Bbbk} A < \infty$, then $A\,\mathrm{fd} = A\,\mathrm{grmod}$ and $A\,\mathrm{qgr} = 0$, a category of single object 0 and $\mathrm{Hom}(0,0) = \{id\}$.

Thm (Serre) Assume A is commutative, $A_0 = \Bbbk$ and A is generated by A_1 as an algebra over $\Bbbk$

Set $X := \mathrm{proj}\, A$ the usual proj scheme of A

Then the functor

$$\Gamma_* : \mathrm{coh}\, X \longrightarrow A\,\mathrm{qgr}, \quad \Gamma_*(\mathcal{F}) := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$$

is an equivalence of categories.

Moreover, $\Gamma_* \circ (\mathcal{O}(1) \otimes_X -) \cong (1) \circ \Gamma_*$,

$$\Gamma_*(\mathcal{O}_X) \cong \pi A.$$

Example $A = \mathrm{Sym}\, V = k[X_0, \cdots, X_n]$

$\deg X_i = 1$ for $i = 0, \cdots, n$.

Then $\mathrm{proj}\, A = \mathbb{P}^n$ and hence

$$\mathrm{coh}\, \mathbb{P}^n \cong A\, \mathrm{qgr}.$$

$\lrcorner$

Lemma (J. Miyachi)

Let $\mathcal{A}$ be an abelian cat and $\mathcal{S} \subset \mathcal{A}$ a Serre sub cat.

Let $D^b_{\mathcal{S}}(\mathcal{A}) \subset D^b(\mathcal{A})$ be a full sub cat of objects M such that $H^n(M) \in \mathcal{S}$ $\forall n \in \mathbb{Z}$.

Then the quotient functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{S}$ induces an equivalence

$$D^b(\mathcal{A}) / D^b_{\mathcal{S}}(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}/\mathcal{S})$$

of triangulated categories.

$\lrcorner$

$\circledcirc$
$$\frac{D^b(A\, \mathrm{grmod})}{D^b_{A\mathrm{fd}}(A\, \mathrm{grmod})} \xrightarrow{\sim} D^b(A\, \mathrm{qgr}).$$

§2

§2.1 singular derived categories.

Let $A\,\mathrm{grperf} := K^b(A\,\mathrm{grproj})$ be the full sub cat of $D^b(A\,\mathrm{grmod})$ consisting of objects M which are represented by bounded complexes of finitely generated graded projective A-modules.

<u>Def</u> (Buchweitz, Orlov).

$$A\,\mathrm{grSing} := \frac{D^b(A\,\mathrm{grmod})}{K^b(A\,\mathrm{grproj})}$$

"the singular derived category of A". ⌟

◎ If $\mathrm{grgl.dim}\, A < \infty$, then $K^b(A\,\mathrm{gr}) = D^b(A\,\mathrm{grmod})$ and hence $A\,\mathrm{grSing} = 0$.

§2.2 the stable category.

For $M, N \in A\,\mathrm{grmod}$, we define a subset $\mathcal{P}(M,N) \subset \mathrm{Hom}_{A\,\mathrm{grmod}}(M,N)$ to be the subset of morphisms $f: M \to N$ that factor through some projective object $\underline{P}$.

$$\mathcal{P}(M,N) := \left\{ f \in \mathrm{Hom}(M,N) \;\middle|\; \begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, dashed] & & N \\ & \underline{P} \arrow[ur, dashed] & \end{tikzcd} \right\}$$

$$\underline{P} \in A\,\mathrm{grproj}.$$

<u>Rmk</u> We obtain a $\Bbbk$-linear sub bifunctor $\mathcal{P}(-,+)$ of $\mathrm{Hom}_{A\,\mathrm{grmod}}(-,+)$.

<u>Def</u> We define a $\Bbbk$-linear category $A\,\underline{\mathrm{grmod}}$ in the following way:

$$\mathrm{ob}(A\,\underline{\mathrm{grmod}}) := \mathrm{ob}(A\,\mathrm{grmod})$$

$$\mathrm{Hom}_{A\,\underline{\mathrm{grmod}}}(M,N) := \frac{\mathrm{Hom}_{A\,\mathrm{grmod}}(M,N)}{\mathcal{P}(M,N)}$$

"the graded stable cat of A".

∘ By the above remark, this definition provides a $\Bbbk$-linear category.

§2.3

Assume that A is finite dimensional and Frobenius.

Then A grmod is a Frobenius category

(i.e. An exact category $\mathcal{C} = (\mathcal{C}, \mathcal{E})$ s.t. adm. proj. obj $\mathcal{P}$ coincides with adm inj obj $\mathcal{I}$.)

Thm (Happel) Let $\mathcal{C}$ be a Frob. cat

Then the stable cat $\underline{\mathcal{C}} = \frac{\mathcal{C}}{[\mathcal{P}]}$ has a str of tri cat.

∘ $A\,\underline{\text{grmod}}$ has a str of tri cat.

Shift ΣM is given by co-syzygy $\Omega^{-1}M$.

$$0 \to M \longrightarrow \underset{\text{inj hull}}{I(M)} \longrightarrow \Omega^{-1}(M) \to 0 \quad (\text{ex})$$

exact tri A triangle $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma M$

is exact if it is isomorphic to a triangle obtained in the following way:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, dashed] & N \arrow[r] \arrow[d, dashed, "h"] & 0 & (\text{ex}) \\ 0 \arrow[r] & L \arrow[r] & I(L) \arrow[r] & \Omega^{-1}(L) \arrow[r] & 0 & (\text{ex}) \end{tikzcd}$$

in A grmod.

- $h \circ g$ factors through $I(L)$, hence $h \circ g = 0$ in A $\underline{\text{grmod}}$.

---

§2.4 The composition

$$A\,\text{grmod} \overset{I}{\hookrightarrow} D^b(A\,\text{grmod}) \xrightarrow{\pi} A\,\text{grSing}$$

sends A grproj to 0.

$$\rightsquigarrow \quad A\,\underline{\text{grmod}} \xrightarrow{\overline{I}} A\,\text{grSing}.$$

Thm (Rickard)

The above functor $\overline{I}$ is an equivalence of tri cat's.

<u>Rmk</u> A generalization to an Iwanaga-Gorenstein algebra ($\mathrm{injdim}_A A < \infty$, $\mathrm{injdim}\, A_A < \infty$) by Buchweitz, Happel.

- $M \in A\,\mathrm{grmod}$ is called Cohen-Macaulay if $\mathrm{Ext}^i_A(M, A(n)) = 0$ for $i > 0$, $n \in \mathbb{Z}$.

(A generalization of Maximal CM modules over a com Gor alg)

- We have $A\,\underline{\mathrm{grCM}} \xrightarrow{\sim} A\,\mathrm{grSing}$.

<u>§3</u>

<u>Thm</u> Let $A = \bigoplus_{i=0}^{n} A_i$ be Koszul Frob alg. Assume $A^!$ is Noetherian. Then the Koszul functor induces equiv

$$A\,\underline{\mathrm{grmod}} \xrightarrow{\sim} D^b(A^!\,\mathrm{qgr}).$$

Proof $F : D^b(A\,\mathrm{gr}) \longrightarrow D^b(A^!\,\mathrm{gr})$

$$A^* \longmapsto A_0$$

A : Frob $\Rightarrow A \cong (A^*)(-n)$.

$F(A) = F(A^*(-n)) = A_0(n)[n]$.

$$\begin{array}{ccc} K^b(A\,\mathrm{grproj}) & = & \mathrm{thick}\{A(n) \mid n \in \mathbb{Z}\} \\ F \downarrow \wr & & \\ D^b_{fd}(A^!\,\mathrm{gr}) & = & \mathrm{thick}\{A^!_0(n) \mid n \in \mathbb{Z}\} \end{array}$$

$\rightsquigarrow \underline{A\,\mathrm{grmod}} \xleftarrow{\sim} A\,\mathrm{grsbg} \xrightarrow{\sim} D^b(A^!_{\S}\,\mathrm{gr})$

Part I Koszul functors

Let $A = \bigoplus_{i \geq 0} A_i$ be Koszul alg.

For $i, j \in \mathbb{Z}$,

$$H^j(\mathbb{R}\mathrm{Hom}_{A_{gr}}(A_0, A_0(-i)[i]))$$

$$= \mathrm{Ext}^{i+j}_{A_{gr}}(A_0, A_0(-i)) = \begin{cases} 0 & i \neq j \text{ or } i < 0, \\ A^!_i & i = j \text{ \& } i \geq 0. \end{cases}$$

Thus

$$\left(\bigoplus_{i \geq 0} \mathbb{R}\mathrm{Hom}_{A_{gr}}(A_0, A_0(-i)[i])\right)^{op} \xrightarrow{\sim} \left(\bigoplus \mathrm{Ext}^i_{A_{gr}}(A_0, A_0(-i))\right)^{op}$$

$$= A^!$$

Define a cpx of bigraded A-$A^!$ bimodules

$$\tilde{A}_0 := \bigoplus_{i \in \mathbb{Z}} A_0(-i)[i]$$

We set

$$K := \mathbb{R}\mathrm{Hom}_{A\text{-}gr}(\tilde{A}_0, -) : D(A\text{-Gr}) \longrightarrow D(A^!\text{Gr})$$

<u>claim</u> K coincides with F on $D^+(A\text{-Gr})$

(i) $\widetilde{A}_0{}^{\circ}_{\bullet,-p} = K(p)[-p] = \bigoplus_{i\geq 0} A\otimes (A^!_i)^*(p-i)[i-p]$

$$K(M)^n_p = \mathrm{Hom}^{\circ}_{A\mathrm{Gr}}(\widetilde{A}_0{}^{\circ}_{\bullet,-p}, M[n])$$

$$= \mathrm{Hom}^{\circ}_{A\mathrm{Gr}}\Big(\bigoplus_{i\geq 0} A\otimes (A^!_i)^*, M(-p+i)[n-i+p]\Big)$$

$$= \prod_i \mathrm{Hom}^{\circ}_{A\mathrm{Gr}}(A\otimes A^{!*}_i, \quad " \quad)$$

$$= \prod_i \mathrm{Hom}^{\circ}_{A_0\mathrm{Gr}}(A^{!*}_i, \quad " \quad)$$

$$= \prod_{i\geq 0} A^!_i \otimes M^{n-i+p}_{-p+i}$$

$$= \bigoplus_{i\geq 0} A^!_i \otimes M^{n-i+p}_{-p+i} \quad \big(\because M \text{ bdd below}\big)$$

$\square$

Similarly G coincides with $\widetilde{A}_0 \overset{L}{\underset{A^!}{\otimes}} -$.

$$\begin{array}{ccc}
\widetilde{A}_0 \overset{L}{\underset{A^!}{\otimes}} - : D(A^!\mathrm{Gr}) & \rightleftarrows & D(A\mathrm{Gr}) : R\mathrm{Hom}(\widetilde{A}_0, -) \\
\cup & & \cup \\
D^{\uparrow}(A^!\mathrm{Gr}) & \overset{\sim}{\rightleftarrows} & D^{\downarrow}(A\mathrm{Gr}) \\
A^! & \longleftrightarrow & A_0 \\
A^!_0 & \longleftrightarrow & A^*
\end{array}$$

Define a full subcat $\underline{LP}_{A^!} \subset D^{\uparrow}(A^!Gr)$ $(\underline{LI}_A \subset D^{\downarrow}(AGr))$ to be the full sub cat of cpx $\underline{M}$ s.t.

for $\forall i$, M^i is dir summand of $A^{\oplus I}(i)$.

(resp. $\underline{M}^i$ of $A^{*\pi I}(i)$).

Prop Koszul functors induces equiv

$$\underline{LP}_{A^!} \xrightarrow{\sim} AGr_{<\infty} \quad \begin{pmatrix}\text{bdd above} \\ \text{gr mod's}\end{pmatrix}$$

$$\underline{LI}_A \xrightarrow{\sim} A^!Gr_{>-\infty} \quad \begin{pmatrix}\text{bdd below} \\ \text{gr mod's}\end{pmatrix}$$

Rmk In general Koszul functors does not give an eq of whole der cat's.

If Koszul functor gives an equivalence $D(A^!Gr) \cong D(AGr)$, then gldim $A < \infty$.