AS-regular algebras and Frobenius algebras

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①

☆ **Thm** [S.P. Smith, '96, Prop 5.10]

(Noncomm alg geometry)

A: Koszul alg (gl dim $A < \infty$)

$\Rightarrow$ [A: AS-regular (alg) $\Leftrightarrow$ $A^!$ (Koszul dual of A) : Frobenius]

("AS-Gorenstein alg")

(Rep theory of fin dim alg)

Notation (due to [Smith])

- k : a field ($\otimes := \otimes_k$)
- $A = \bigoplus_{i \geq 0} A_i$: connected graded locally finite non-negatively graded k-alg.

(簡単のために)

(※ $\forall i$, $\dim_k A_i < \infty$)

$\left(A_{\geq 1} := \bigoplus_{i \geq 1} A_i\right)$ $\quad$ $\left(A_0 = k_A = A/A_{\geq 1}\right)$

(trivial module)

- GrMod$A$ (Gr-A) : the cat of graded right A-module.
- $\mathrm{Hom}_{\mathrm{Gr}}(N, M)$: the morph in GrMod$A$.

$\quad = \{f : N \to M : A\text{-hom} \mid f(N_i) \subset M_i, \forall i \in \mathbb{Z}\}$

- $\mathrm{Hom}_A(N, M)_d := \{f : N \to M : A\text{-hom} \mid f(N_i) \subset M_{i+d} \ (\forall i \in \mathbb{Z})\}$

($\hom_A(N, M(d))$) $\quad$ the degree of f is d.

$\underline{\mathrm{Hom}}_A(N, M) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_A(N, M)_d$.

(※ Note that, N: fin gen $\Rightarrow$ $\underline{\mathrm{Hom}}_A(N, M) = \mathrm{Hom}_A(N, M)$)

- ($i \in \mathbb{Z}$) We write $\underline{\mathrm{Ext}}^i_A(N, M)$ for the derived functor of $\underline{\mathrm{Hom}}_A(N, M)$

$\underline{\mathrm{Ext}}^*_A(N, M) := \bigoplus_{p \in \mathbb{Z}} \underline{\mathrm{Ext}}^p_A(N, M)$. $\quad$ $\underline{\mathrm{Ext}}^*_A(N, M) = \mathrm{Ext}^*_A(N, M)$

※ (説明なし)

- $M(1) := M$ (as a right A-module) : the shif functor (1) on GrMod$A$. but with grading $M(1)_i := M_{i+1}$.

Thus, $\underline{\mathrm{Hom}}_A(N, M) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}}(N, M(n))$.

- $M \in \mathrm{GrMod}A$, $M^* := \underline{\mathrm{Hom}}_k(M, k)$

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§1. Preliminaries

§2. AS-reg alg and Frobenius alg, Smith's Thm ($\to$ examples)

§1

- A graded A-module $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda$ is called bounded below if $M_\lambda = 0$ for $\lambda \ll 0$.

- We recall that the following Nakayama type Lemma.

Lemma 1.1 M: graded A-module bounded below. TFAE.
(1) $A_{\geq 1} M = M$.
(2) $M = 0$.

Recall A graded free A-module P is a graded module of the form $P = A \otimes_k V = \bigoplus_{\lambda \in \Lambda} A(n_\lambda)$, where Λ: an index set, $V := \bigoplus_{\lambda \in \Lambda} k(n_\lambda)$.

Using Lemma 1.1, we can deduce the following.

Prop 1.2 $^\forall M$: graded A-module bounded below.
$\Rightarrow$ $^\exists$ a surjective graded A-module hom $f: F \to M$ (: free) s.t. $\operatorname{Ker} f \subset A_{\geq 1} F$.
Moreover, such morph.s are unique up to iso.
(※) (Namely, $^\exists$ a surjective hom $f': F' \to M$ of graded A-hom s.t. $\operatorname{Ker} f' \subset A_{\geq 1} F'$ $\Rightarrow$ $^\exists$ an iso $g: F \to F'$ s.t. $f'g = f$.)

Cor. 1.3 Every graded projective A-module P bounded below is free.
(※ $\boxed{\text{free} \Leftrightarrow \text{proj}}$)

- We recall mini proj res of a graded A-module.

Def 1.4 M: graded A-module of bounded below. A graded proj res
$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$$
is called minimal, if $\operatorname{Ker} \varepsilon \subset P_0$, $\operatorname{Ker} d_n \subset A_{\geq 1} P_n$ for all $n \geq 1$.

- By Prop 1.2, $^\forall M$: graded A-module bounded below, M has a mini proj res whose term P_n is a graded free module bounded below.
- By Cor 1.3, each term P_n is of the form

$$\cdots \to A\otimes_k V_n \xrightarrow{d_n} A\otimes_k V_{n-1} \to \cdots \to A\otimes_k V_1 \xrightarrow{d_1} A\otimes_k V_0 \xrightarrow{\varepsilon} M \to 0 . \quad \cdots (1.1)$$

※ ($\overset{※}{\text{L}}$inear free resolution is minimal)

(紹介しない) $A := k[x_1, \dots, x_n]$: polynomial ring ($\deg x_i = 1$)

$$0 \to \oplus A(-n) \xrightarrow{M^n\cdot} \oplus A(-n+1) \xrightarrow{M^{n-1}\cdot} \cdots \xrightarrow{M^2\cdot} \oplus A(-1) \xrightarrow{M^1\cdot} \oplus A \to k \to 0$$

where $M^l := (m^l_{ij})$, $m^l_{ij} \in A_1$.

This is mini free res of k as graded left A-modules.

(紹介しない) $A := k\langle x, y\rangle / (\alpha xy - yx)$ ($\alpha \in k\setminus\{0\}$) ($\mathrm{gl.dim}\, A = 2$, $H_A(t) = \frac{1}{(1-t)^2}$ (Hilbert series of A))

$$0 \to A(-2) \xrightarrow{\binom{\alpha y}{-x}\cdot} A(-1)^{\oplus 2} \xrightarrow{(x,y)\cdot} A \to k \to 0$$

This is mini free res $\Leftrightarrow \alpha \neq 0$

Def. 1.5 $M \in A\text{-}\mathrm{gr}_{\mathbb{Z}^n} \subseteq A\text{-}\mathrm{gr}_{\mathbb{Z}}$

M has a linear resolution

$\overset{\text{def}}{\Leftrightarrow}$ $^\exists V^{(0)}, V^{(1)}, \dots, V^{(a)}$: k-vec sp, $^\exists a \in \mathbb{Z}$ s.t. $\cdots \to A\otimes_k V^{(1)} \to A\otimes_k V^{(0)} \to M \to 0$, $V^{(i)} = (V^{(i)})_{i+a}$.

Rmk
- Linear free resol is minimal. (A_0 has a linear free resolution)
- (Ex) A: Koszul, the Koszul resolution of A_0 is a linear free resolution.

Prop 1.6 M : a graded A-module of bounded below. We take a mini proj res of M as give in the above (1.1). Then the following hold.

(1) $\underline{\mathrm{Ext}}^p_A(M, k) \cong V_p^*$

(2) $\underline{\mathrm{Tor}}^A_p(k, M) \cong V_p$.

Lemma 1.7 $A = \bigoplus_{i\geq 0} A_i$: connected locally finite non-negatively graded alg.

Then, for $n \in \mathbb{N}$, the following statements are equivalent.

(1) $\mathrm{gl.dim}\, A \leq n$.

(2) $\underline{\mathrm{Ext}}^{n+1}_A(k, k) = 0$.

§2

Def. 2.1 R: a fin dim k-alg.

R: <u>Frobenius alg</u> $\overset{\text{def}}{\Longleftrightarrow}$ $R \cong R^*$ as left R-modules

where $R^* := \mathrm{Hom}_k(R, k)$ is given by the left module str induced by the right regular action of R on its self.

- A Frobenius alg is injective as a module over its self.

Lemma 2.2 R: a fin dim, connected graded k-alg s.t. $R_n \neq 0$ but $R_{\geq n+1} = 0$. TFAE.

(1) R: Frobenius

(2) $R \cong R^*(-n)$ as graded R-modules.

- If R is in Lemma 2.2, by the hypothesis of connectedness, then $\exists !$ isomorphism $R \longrightarrow R^*(-n)$ up to scalar multiples.

Def. 2.3 [Artin-Schelter, 1987] $A = \bigoplus_{i \geq 0} A_i$: connected graded k-alg. (locally finite)

A: n-dim <u>AS-regular alg</u> $\overset{\text{def}}{\Longleftrightarrow}$
(i) $\mathrm{gl.dim}\, A = n < \infty$,
(ii) $\mathrm{GKdim} < \infty$, (Gelfand-Kirillov dimension)
(iii) $\mathrm{Ext}^i_A(k, A) \cong \begin{cases} k, & i = n \\ 0 & i \neq n \end{cases}$ (Gorenstein condition)

※ $\mathrm{GKdim}\, A := \inf\{ d \in \mathbb{R}^+ \mid \exists c \gg 0, \forall i \gg 0, \dim_k A_i \leq c i^{d-1} \}$

This is a non-comm analogue of Krull dimension.

(if $H_M(t) = f(t)(1-t)^{-r}$, where $r \geq 0$, $t \in \mathbb{Z}[t, t^{-1}]$, and $f(1) \neq 0$, then $\mathrm{GKdim}(M) = r$.)

Ex 2.4 (AS-reg alg)

(1) $A := k[x_1, \dots, x_n]$ (polynomial ring) : comm n-dim AS-reg alg.

※ (4) $A := k\langle x, y \rangle / (yx)$ [AS]
$\mathrm{gldim}\, A = 2$, $\mathrm{GKdim}\, A = 2$
but (iii) does not hold.

(2) $A := k\langle x_1, \dots, x_n \rangle / (x_j x_i - \alpha_{i,j} x_i x_j)$ $(1 \leq i < j \leq d,\ \alpha_{i,j} \in k \setminus \{0\})$ ($\deg x_i = 1$)
(free alg) (skew polynomial alg) : n-dim AS-reg alg

※ (2), (3) ($k = \bar{k}$, $\mathrm{char}\, k = 0$)

(3) $A := k\langle x, y, z \rangle / (yx - \alpha z^2, zy - \beta x^2, xz - \gamma y^2)$ $(\alpha, \beta, \gamma \in k \setminus \{0\},\ \alpha\beta\gamma \neq 0, 1)$ ($\deg x = \deg y = \deg z = 1$)
: 3-dim AS-reg alg (※ Koszul)

$E = \mathcal{V}(xyz)$

※ $A = \mathcal{A}(E, \sigma)$ (Type S_3 : $E \circlearrowleft^{\sigma} \subset \mathbb{P}^2$, σ circulates three component.)

Λ

(i)

(ii)

(iii)

$$0 \to \bigoplus A(-n) \xrightarrow{M^d \cdot} \bigoplus A(-n+1) \xrightarrow{M^{d-1} \cdot} \cdots \xrightarrow{M^2 \cdot} \bigoplus A(-1) \xrightarrow{M^1 \cdot} \bigoplus A \to {}_A k \to 0$$

$M^l = (m_{ij}^l)$ $m_{ij}^l \in A_1$

$\mathrm{Hom}_A(-, A)$

$$0 \leftarrow k(d) \leftarrow \bigoplus A(d) \xleftarrow{\cdot M^d} \bigoplus A(d-1) \cdots \xleftarrow{\cdot M^2} \bigoplus A(1) \leftarrow \bigoplus A \leftarrow 0$$

$d = n$

Gorenstein mini free res of $k(d)$ as graded left A-module.

'χ' cond $\to$ (kのmini free res が左右対称になっていることをいっている)

- From now on, all alg.s are quadratic (※ and therefore connected.)

※ **Def.** A: graded k-alg

- A: quadratic $\overset{\text{def}}{\iff}$ $A = T(V)/(R)$ where
  - (i) V: fin dim k-vec sp, concentrated in deg 1.
  - (ii) $T(V)$: the tensor alg on V, with the induced grading.
  - (iii) (R): the ideal of $T(V)$ generated by a subset $R \subset V \otimes_k V$.
- The dual of a quad alg A is $A^! := T(V^*)/(R^\perp)$ where $R^\perp := \{\lambda \in V^* \otimes_k V^* \mid \lambda(v) = 0, \forall v \in R\}$.
- We identify $(V \otimes_k V)^*$ with $V^* \otimes_k V^*$ by defining $(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v)$, $\forall \alpha, \beta \in V^*$, $\forall u, v \in V$.

Rmk If, in def of $A^!$, we identify $(V \otimes_k V)^*$ with $V^* \otimes_k V^*$ by defining $(\alpha \otimes \beta)(u \otimes v) = \alpha(v) \otimes \beta(u)$, $\forall \alpha, \beta \in V^*$, $\forall u, v \in V$, then $\mathrm{Ext}^*_A(k,k)$ is isomorphic to $(A^!)^{op}$.

※ **Rmk** $\{x_\lambda\}$: basis for A_1, $\{\xi_\lambda\}$: its dual basis in A_1^*

$\Rightarrow$ $e := \sum_\lambda x_\lambda \otimes \xi_\lambda \ (\in A_1 \otimes (A_1^!)^*)$ is independent of the choice of basis.

Thm 2.5 (Recall) A: quad alg. TFAE.

(1) A: Koszul. (2) A^{op}: Koszul

(3) $A^!$: Koszul

(4) $\mathrm{Ext}^*_A(k,k) \cong (A^!)$ as a graded k-alg.s

(5) The Koszul complex is a mini res of k $\left(\overset{\shortparallel}{A_0}\right)$ ← A: connected

※ ((6) $\mathrm{Ext}^p_A(k,k)_j = 0$ if $p+j \neq 0$.)

(※ In this case, $A^!$ is called Koszul dual)

Thm 2.6 A: Koszul. TFAE.
(1) $\mathrm{gl.dim}\, A < \infty$ (2) $\dim_k A^! < \infty$.

(※ $A = \bigoplus_{i \geq 0} A_i$: connected locally finite non-negatively graded alg.)

proof) Since $A^!$ is locally finite, the condition (2) is equivalent to the condition (2′) $A^!_p = 0$ ($\forall p \gg 0$).

(1) ⇒ (2′) By Thm 2.5 and Lemma 1.7. $A^!_p \underset{(\text{Thm } 2.5)}{\cong} \mathrm{Ext}^p_A(k, k) \underset{(\text{Lemma } 1.7)\,((1)\, \mathrm{gl.dim}\, A < \infty)}{=} 0$ for $p \gg 0$.

(2′) ⇒ (1) It follows from Lemma 1.7. ▨

※ (1) $\mathrm{gl.dim}\, A \overset{= n}{<} \infty$ $\overset{(\text{Lemma } 1.7)}{\Updownarrow}$ $\mathrm{Ext}^{n+1}_A(k, k) = 0 \underset{(\text{Thm } 2.5)}{\Longleftrightarrow} A^!_{n+1} = 0 \overset{(2')}{\Leftarrow} A^!_p = 0$ ($\forall p \gg 0$) $(\exists n+1)$

☆ Thm 2.7 ([Smith, '96, Prop 5.10])
A: Koszul alg
⇒ [A: AS-regular alg ⇔ $A^!$ (Koszul dual of A): Frobenius]

(Rmk Without assuming "Koszul", the similar statement holds by [Lu-Palmieri-Wu-Zhang, Cor D].)

proof) ※(Note that since $\underset{\substack{\| \\ n}}{\mathrm{gl.dim}\, A} < \infty$, by Thm 2.6, $A^!$ is finite dim. ($\dim_k A^! < \infty$)) (⑦の☆へ回す)

The groups $\mathrm{Ext}^i_A(k, A)$ are the homology groups of the complex obtained by applying $\mathrm{Hom}_A(-, A)$ to the Koszul complex for A, that is, $\mathrm{Ext}^i_A(k, A)$ are the homology group of the complex

$$0 \to A_A \xrightarrow{d} A^!_1 \otimes_k A_A \xrightarrow{d} \cdots \xrightarrow{d} A^!_n \otimes_k A_A \to 0 \quad \cdots (*)$$

(of right A-module, where the differential d is left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. (※ $\{x_\lambda\}$: basis for A_1, $\{\xi_\lambda\}$: its dual basis in A_1^*))

∴ $\boxed{A\text{: AS-reg alg}}$ ⇔ (*) is exact except at final position where its homology is $k_A(n)$.

Def (· $\mathrm{gl.dim}\, A = n$ · $\mathrm{Ext}^i_A(k, A) \cong \begin{cases} k & (i = n) \\ 0 & (i \neq n) \end{cases}$)

⑥ 前半の(*)の理由

A: Koszul

$$\Rightarrow \quad 0 \to K_n(A) \to \cdots \to K_1(A) \to K_0(A) \to 0 \quad \text{(※ left } A\text{-module)}$$

(gl.dim $A < \infty$) ↑

$K_n(A) = A \otimes_k (A^!_n)^*$, $K_1(A) = A \otimes_k (A^!_1)^*$

$\left(K_0(A) = A \otimes_k (A^!_0)^* = A \otimes_k k = A \right) \leftarrow (A_0 = k)$

: Koszul complex for A

↓ Applying $\underline{\mathrm{Hom}}_A(-, A)$ $\left(※ \ \underline{\mathrm{Hom}}_A(A \otimes_k (A^!_n)^*, A) \cong A^!_n \otimes_k A \ (\forall n) \right)$

$$0 \to A_A \xrightarrow{d} A^!_1 \otimes_k A_A \xrightarrow{d} \cdots \xrightarrow{d} A^!_n \otimes_k A_A \to 0 \quad \cdots (*)$$

$\left(\underline{\mathrm{Hom}}_A(A, A) \cong \right)$ (right A-module) $A_A = \left(A^!_0 \otimes_k A_A \right)$

$\boxed{\underline{\mathrm{Ext}}^i_A(k, A) \text{ are the homology group of the complex } (*)}$

- By Thm 2.5 $\left((2)\Leftrightarrow(5)\right)$, A^{op} is Koszul $\left(\Updownarrow \text{ A is Koszul}\right)$, so k_A has a mini res given by the Koszul complex:

$$0 \to (A^!_n)^* \otimes_k A \xrightarrow{\delta} \cdots \xrightarrow{\delta} (A^!_1)\otimes_k A \xrightarrow{\delta} A \to k_A \to 0 \quad \cdots (**)$$

(where δ is left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$.)

$\therefore$ $\boxed{A: \text{AS-reg alg}}$ $\Leftrightarrow$ $(*)$, $(**)$ are isomorphic as complex of right A-modules.

$$\left(\begin{tikzcd}
(*):\ 0 \arrow[r] & \underbrace{A^!_0 \otimes_k A}_{(=A)} \arrow[r, "d"] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & A^!_1 \otimes_k A \arrow[r, "d"] \arrow[d, dashed, "*"] & \cdots \arrow[r, "d"] \arrow[dr, phantom, "\circlearrowleft"] & A^!_n \otimes_k A \arrow[r] \arrow[d, dashed, "*"] \arrow[dr, dashed] \arrow[r, phantom, "\circlearrowleft"] & 0 & \\
(**):\ 0 \arrow[r] & (A^!_n)^* \otimes_k A \arrow[r, "\delta"] & (A^!_{n-1})^* \otimes_k A \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & \underbrace{(A^!_0)^* \otimes_k A}_{(=A)} \arrow[r] & k_A \arrow[r] & 0
\end{tikzcd}\right)$$

$(*) \overset{\delta}{\downarrow} (**)$: $\left(\boxed{\Phi \otimes 1_A}\right)$, $A^!_0$ $(=k)$

$\therefore$ $\boxed{A: \text{AS-reg alg}}$ $\Leftrightarrow$ $\exists$ iso $\Phi: A^! \to (A^!)^*(-n)$ of graded vector sp s.t. $\boxed{\Phi\otimes 1_A}$ is an iso of complexes.

that is, s.t. $\delta\circ(\Phi\otimes 1_A) = (\Phi\otimes 1_A)\circ d$.

Given the above descriptions of d and δ, if $\alpha\otimes a \in A^!_i \otimes_k A$, then

$$\begin{cases} (\delta\circ(\Phi\otimes 1_A))(\alpha\otimes a) = \sum_\lambda \xi_\lambda \Phi(\alpha)\otimes x_\lambda a \\ ((\Phi\otimes 1_A)\circ d)(\alpha\otimes a) = \sum_\lambda \Phi(\xi_\lambda a)\otimes x_\lambda a \end{cases}$$

※ $\left(\delta\left(\sum_\lambda \Phi(\alpha)\otimes a\right) \overset{(\delta \text{ def})}{=} \sum_\lambda \xi_\lambda \Phi(\alpha)\otimes x_\lambda a\right)$

※ $\left(((\Phi\otimes 1_A)\circ d)(\alpha\otimes \alpha) = \sum_\lambda \Phi(\xi_\lambda a)\otimes x_\lambda a\right)$

$\therefore$ $\boxed{A: \text{AS-reg alg}}$ $\Leftrightarrow$ $\exists$ iso $\Phi: A^! \to (A^!)^*(-n)$ of graded vec sp s.t. $\xi_\lambda \Phi(\alpha) = \Phi(\xi_\lambda \alpha)$, $\forall \alpha\in A^!$, $\forall\lambda$.

This precisely means that Φ is a left $A^!$-module iso.

(Recall that $A^!$ is finite dim.) (★ ⑥の $\dim_k A^! < \infty$ はここで説明する。) $\left(※ + \frac{A^!_n \neq 0,}{A^!_{\geq n+1} = 0}\right)$

So, by Lemma 2.2, Φ is equivalent to the condition that $A^!$ is Frobenius. ▨

Ex 2.8

(1) $A := k\langle x_1, \cdots, x_n\rangle / I$, $I := \langle \alpha_j x_i x_j - \alpha_i x_j x_i \mid \alpha_i, \alpha_j \in k \setminus \{0\},\ 1 \le i < j \le n \rangle$

: the skew polynomial alg

: (n-dim AS-reg alg, Koszul).

By calculation, $A^!$: Koszul dual of A.

$$A^! = k\langle x_1, \cdots, x_n\rangle / I', \quad I' := \langle x_i^2,\ x_i x_j + \alpha_{ij} x_j x_i \rangle$$

: the quantum exterior alg. $\left(\alpha_{ij} := \dfrac{\alpha_j}{\alpha_i}\right)$

$\therefore$ By Thm 2.7 (Smith's Thm), $A^!$ is Frobenius alg.

(2) $A := k\langle x, y, z\rangle / (yx - \alpha z^2,\ zy - \beta x^2,\ xz - \gamma y^2)$, $\left(\begin{array}{l}\alpha, \beta, \gamma \in k\setminus\{0\}, \\ \alpha\beta\gamma \neq 0, 1\end{array}\right)$

: 3-dim AS-reg alg, Koszul.

By calculation, $A^!$: Koszul dual of A.

$$A^! = k\langle x, y, z\rangle \Big/ \begin{pmatrix} x^2 + \beta zy,\ xy, \\ y^2 + \gamma xz,\ yz, \\ z^2 + \alpha yx,\ zx \end{pmatrix}$$

$(\alpha, \beta, \gamma \in k\setminus\{0\})$
$(\alpha\beta\gamma \neq 0, 1)$.

$\therefore$ By Thm 2.7 (Smith's Thm), $A^!$ is Frobenius alg.

Ex (2) $A = k\langle x,y,z\rangle \Big/ (\overbrace{yx-\alpha z^2}^{f_1},\ \overbrace{zy-\beta x^2}^{f_2},\ \overbrace{xz-\gamma y^2}^{f_3})$

: 3-dim Koszul AS-reg alg.

$(\alpha\beta\gamma \neq 0, 1)$

$(\alpha, \beta, \gamma \in k \setminus \{0\})$

$A^!$: Koszul dual of A

$A^! = k\langle X,Y,Z\rangle / I'$ $\quad \left(I' = \{ F \in \underset{\displaystyle (k\langle x,y,z\rangle_2)^*}{\overset{}{k\langle X,Y,Z\rangle_2 \atop \shortparallel}} \mid F(f_i) = 0 \ (i=1,2,3) \} \right)$

X, Y, Z : dual basis of x, y, z

$F := b_{11}X^2 + b_{12}XY + b_{13}XZ + b_{21}YX + b_{22}Y^2 + b_{23}YZ + b_{31}ZX + b_{32}ZY + b_{33}Z^2$ とおき, $(b_{ij} \in k)$

$F(f_1) = 0$, $F(f_2) = 0$, $F(f_3) = 0$

となる b_{ij} を求める.

$\therefore$ $F(f_1) = b_{21} - \alpha b_{33} = 0$, $F(f_2) = b_{32} - \beta b_{11} = 0$, $F(f_3) = b_{13} - \gamma b_{22} = 0$.

$\therefore$ $b_{21} = \alpha b_{33}$, $b_{32} = \beta b_{11}$, $b_{13} = \gamma b_{22}$

$$\begin{aligned}
\therefore\ F &= b_{11}X^2 + b_{12}XY + (\gamma b_{22})XZ + (\alpha b_{33})YX + b_{22}Y^2 + b_{23}YZ + b_{31}ZX \\
&\quad + (\beta b_{11})ZY + b_{33}Z^2 \\
&= b_{11}(X^2 + \beta ZY) + b_{12}XY + b_{22}(\gamma XZ + Y^2) + b_{33}(\alpha YX + Z^2) \\
&\quad + b_{23}YZ + b_{31}ZX.
\end{aligned}$$

$\therefore$ $I' = (X^2 + \beta ZY,\ \gamma XZ + Y^2,\ \alpha YX + Z^2,\ XY,\ YZ,\ ZX)$.

$$\therefore\ A^! = k\langle X,Y,Z\rangle \Big/ \left(X^2 + \beta ZY,\ \underset{}{\overset{(XZ+\gamma' Y^2)}{\overset{\shortparallel}{\gamma XZ + Y^2}}},\ \overset{(YX+\alpha' Z^2)}{\overset{\shortparallel}{\alpha YX + Z^2}},\ XY,\ YZ,\ ZX \right)$$

$\left(\gamma' = \frac{1}{\gamma},\ \alpha' = \frac{1}{\alpha}\right)$

$(\alpha\beta\gamma \neq 0, 1)$

$(\alpha, \beta, \gamma \in k \setminus \{0\})$.

$\boxed{\text{(1)も同様の計算}}$ でできる.