Graded algebra and quadratic algebras, and Hilbert series

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Notations
§1. K-dual
§2. Filtrations on modules
§3. Unicity of gradings
§4. Quadratic ring
§5. Symmetric v.s. exterior algebras 1.

Notations

k: field
K: semisimple ring
For a ring A,
A-Mod: the category of all left A-modules.
(Mod-A) (right)
A-mod: —〃— finitely generated left —〃—.
(mod-A) (right)

For a graded ring A,
A-Gr: the cat. of all $\mathbb{Z}$-graded left A-modules.
(Gr-A) (right)
A-gr: —〃— f.g. $\mathbb{Z}$-graded left —〃—.
(gr-A): (right)

$[s]$: Homological shift (For a complex X, $X[s]^i = X^{s+i}$

$X: \cdots \to X^{-1} \to X^0 \to X^1 \cdots$
$X[1]: \cdots \to X^0 \to X^1 \to X^2 \cdots$

(s): Grading shift (For a graded module M, $M(s)_i = M_{s+i}$

M	$M(1)$
$\vdots$	$\vdots$
M_{-1}	M_0
M_0	M_1
M_1	M_2
$\vdots$	$\vdots$

	Morphisms	Extensions
A-Mod	Hom_A	Ext_A
Mod-A	Hom_{-A}	Ext_{-A}
A-Gr	hom_A	ext_A
Gr-A	hom_{-A}	ext_{-A}

§1 K-dual

$V \in K\text{-Mod}$, $W \in \text{Mod-}K$,

Def 1 (K-dual)

$$\underset{\substack{\Uparrow \\ \text{Mod-}K}}{V^*} := \mathrm{Hom}_K(V, K), \quad \underset{\substack{\Uparrow \\ K\text{-Mod}}}{{}^*W} := \mathrm{Hom}_{-K}(W, K)$$

Fact 2

- V, W: f.g. $\Rightarrow V \cong {}^*(V^*)$, $W \cong ({}^*W)^*$

- $V_1 \subset V$, $W_1 \subset W$: sub module $\Rightarrow$ $V_1^{\perp} := \{f \in V^* \mid f(V_1) = 0\} \subset V^*$, ${}^{\perp}W_1 := \{g \in {}^*W \mid g(W_1) = 0\} \subset {}^*W$: sub module

$U \in K\text{-Mod-}K \; (= K \underset{\mathbb{Z}}{\otimes} K^{op}\text{-Mod})$

- U^*, ${}^*U \in K\text{-Mod-}K$.
- $U_1 \subset U$ is a sub-bimodule $\Rightarrow$ $U_1^{\perp} \subset U^*$, ${}^{\perp}U_1 \subset {}^*U$ are sub-bimodules

$\otimes := \underset{K}{\otimes}$

- $V \in K\text{-mod-}K$, $U \in K\text{-mod-}K$ $\Rightarrow V^* \otimes U^* \cong (U \otimes V)^*$ as right K-modules. K-bimodules

- $U_1, \dots, U_n \in K\text{-mod-}K$
  $\Rightarrow U_1^* \otimes \cdots \otimes U_n^* \cong (U_n \otimes \cdots \otimes U_1)^*$ as K-bimodules.

- $V, V' \in K\text{-mod} \Rightarrow V^* \otimes V' \cong \mathrm{Hom}_K(V, V')$

A : ring
$X \in A\text{-Mod-}K$
$M \in \text{Mod-}A$
$N \in \text{Mod-}K$

$$\mathrm{Hom}_{-A}(M, \mathrm{Hom}_{-K}(X, N)) \cong \mathrm{Hom}_{-K}(M \underset{A}{\otimes} X, N).$$

Suppose
- $K = \bigoplus_i K_i$: graded ring , $K_i = \begin{cases} K & i=0 \\ 0 & i\neq 0 \end{cases}$
- A : graded ring.

Def3 (Graded K-dual) $V \in K\text{-}Gr$, $W \in Gr\text{-}K$

$V^{\circledast} = \bigoplus_i V_i^{\circledast} := \bigoplus_i (V_{-i})^* \in Gr\text{-}K$. ${}^{\circledast}W = \bigoplus_i {}^{\circledast}W_i := \bigoplus_i {}^*(W_{-i}) \in K\text{-}Gr$.

Fact4 $M \in Gr\text{-}A$, $X \in A\text{-}Gr\text{-}K$.

- ${}^{\circledast}X \in K\text{-}Gr\text{-}A$
- $hom_{-A}(M, {}^{\circledast}X) \cong {}^*((M \otimes_A X)_0)$
- X : projective obj. of A-Mod $\Rightarrow$ ${}^{\circledast}X$: injective obj. of $Gr\text{-}A$

§2 Filtrations on modules.

Def5 A : ring, $M \in A\text{-}Mod$
- $\mathrm{soc}\,M$: the biggest semisimple submodule.
- The socle filtration on M

$$\mathrm{soc}^i M := \begin{cases} 0 & (i=0) \\ \pi : M \twoheadrightarrow M/\mathrm{soc}^{i-1}M. & \\ \mathrm{soc}^i M = \pi^{-1}\,\mathrm{soc}(M/\mathrm{soc}^{i-1}M) & (i \geq 1) \end{cases}$$

Assume that M has finite length,

- $\mathrm{rad}\,M := \bigcap \{ N \subset M \mid M/N : \text{semisimple} \}$.

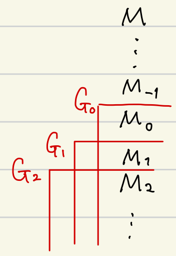
The radical filtration on M.

$$\mathrm{rad}^i M := \begin{cases} M & (i=0) \\ \mathrm{rad}^i M := \mathrm{rad}(\mathrm{rad}^{i-1} M) & (i \geq 1) \end{cases}$$

Suppose $A = \bigoplus_{i \in \mathbb{Z}} A_i$ = positively graded ring (i.e. $A_i = 0$ for $i < 0$)

For $M \in A\text{-}Gr$,

$$G_j M := \bigoplus_{i \geq j} M_i \ : \text{grading filtration.}$$



$G_{j+1} \subseteq G_j$

Rem A_0 is semisimple $\Rightarrow$ $G_j M / G_{j+1} M$: semisimple.

Prop 6 A: positively graded ring.
M: a graded A-module of finite length

Suppose that (A1) A_0 is semisimple
(A2) A is generated by A_1 over A_0.
$\Leftrightarrow A_{>0} = A A_1$

If $M/\mathrm{rad}\, M$ is simple, then $\mathrm{rad}^i M = G_i M$
up to shift.

$(\mathrm{soc}\, M)$ $(\mathrm{soc}^i M = G_{-i} M)$

$\because$) Put $I := A_{>0}$
Since M has finite length, $I^n M = 0$ for $n >> 0$.

Thus $IN = 0$ for any simple subquotient N of M.

$\therefore$ $\mathrm{soc}\, M = \{ m \in M \mid I m = 0 \}$, $\mathrm{soc}^i M = \{ m \in M \mid I^i m = 0 \}$
$\mathrm{rad}\, M = IM$, $\mathrm{rad}^2 M = I^2 M$.

By (A2), $I^i = A_{\geq i}$.

$\therefore$ $\mathrm{soc}^i M = \{ m \in M \mid A_{\geq i} m = 0 \}$
$\mathrm{rad}^i M = A_{\geq i} M$

Suppose $M/\mathrm{rad}\, M$ is simple. i.e. concentrated in one degree j.
$\mathrm{soc}\, M$

Then $M = AM_j$

$\forall \underset{\neq 0}{m} \in M_{j-\nu}$, $\exists a \in A_\nu$ s.t. $\underset{\neq 0}{am} \in M_j$

$\therefore \operatorname{rad}^i M = A_{\geq i} M = A_{\geq i} M_j = \bigoplus_{n \geq i+j} M_n = G_{i+j} M = G_i \left(M(j) \right)$

$G_{i+j}M$

M

$G_{i+j}M$ $\vdots$

M_{i+j}

M_{i+j+1}

$\therefore \bigoplus_{n > j-i} M_n = \{ m \in M \mid A_{\geq i} m = 0 \} = \operatorname{soc}^i M$

$\shortparallel$

$G_{j-i+1} M_n = G_i \left(M(j-2i+1) \right)$

§3 Unicity of gradings.

Prop7 Let A be a positively graded ring satisfying (A2).

Put $I := A_{>0}$. Then $A \cong \bigoplus_i I^i/I^{i+1}$ as a graded ring ($I^0 = A$)

Cor8 Let A be an artinian ring.

Any two grading on A satisfying (A2) are isomorphic as a graded ring.

Let A: graded ring,

$F : A\text{-Gr} \longrightarrow A\text{-Mod}$: forgetful functor.

Def9 $\tilde{M} \in A\text{-Gr}$ is a graded lift of $M \in A\text{-Mod}$

$\Leftrightarrow F\tilde{M} \cong M$.

Prop 10 $M \in A\text{-Mod}$ is indecomposable and of finite length.

If M admits a lift $\tilde{M}$,

then $\tilde{M}$ is unique up to grading shift and isomorphism.

$\because$) Since M is indec., $\mathrm{End}_A(M)$ is local

i.e. non-automorphisms form an ideal.

$\tilde{M}'$ is another lift of M, we have $\mathrm{Hom}_A(M, M) \cong \bigoplus_n \mathrm{hom}_A(\tilde{M}', \tilde{M}(n))$

$$\begin{array}{ccc} \mathrm{id} & \longmapsto & \sum \mathrm{id}_n \end{array}$$

Since id is an automorphism, $\exists 1\ \mathrm{id}_n$ s.t. $F(\mathrm{id}_n) = \mathrm{id}$.

$\therefore\ \mathrm{id}_n : \tilde{M}' \xrightarrow{\sim} \tilde{M}(n)$

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§4. Quadratic ring.

Def 11

- A positively graded ring A is a <u>quadratic ring</u>.

$\overset{\text{def}}{\Leftrightarrow}$ (Q1) A_0 is semisimple
$(A1) =$

(Q2) A is generated over A_0 by A_1 <u>with relations of degree two.</u> (∗)

(∗) { Let $T_{A_0}A_1 := A_0 \oplus A_1 \oplus (A_1 \bigotimes_{A_0} A_1) \oplus \cdots = \bigoplus_{i \geq 0} A_1^{\otimes i}$

$$F : T_{A_0}A_1 \twoheadrightarrow A$$
$$a_1 \otimes \cdots \otimes a_i \longmapsto a_1 \times \cdots \times a_i$$

Then $\operatorname{Ker} F$ is genrated by $R := \ker F \cap (A_1 \bigotimes_{A_0} A_1)$
as a two-sided ideal.
i.e. $\operatorname{Ker} F = (R)$

$$\therefore\ T_{A_0}A_1 / (R) = A.$$

- A graded ring $A = \bigoplus_i A_i$ is <u>left locally finite</u>
(<u>right ⁗</u>)

$\Leftrightarrow$ all A_i are f.g. as left A_0-modules.
(right)

Ex k: field, $V \in k\text{-mod}$

- $T_k V := k \oplus V \oplus (V \otimes V) \oplus \cdots = \bigoplus V^{\otimes i}$

$T_k V$ is quadratic. $\because)$ $R = 0$.

- $R := \{ v \otimes w - w \otimes v \mid v, w \in V \}$

$S(V) := T_k V / (R)$: symmetric algebra

$\cong k[x_1, \ldots, x_{\dim_k V}]$

- $R := \{ v \otimes v \mid v \in V \}$

$\Lambda(V) := T_k V / (R)$: exterior algebra.

Def 12 (Path algebra)

$Q = (Q_0, Q_1)$: quiver (有向グラフ)

Q_0 ← vertices, Q_1 ← arrows

For $\alpha_1, \cdots, \alpha_\ell \in Q_1$,

$\alpha_1 \cdots \alpha_\ell$ is a <u>path of length ℓ</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\exists \; \bullet \xleftarrow{\alpha_1} \bullet \leftarrow \cdots - \bullet \xleftarrow{\alpha_\ell} \bullet$ in Q

※ For $i \in Q_0$, i is a path of length zero and we denote by e_i.

$Q_\ell :=$ the set of all paths of length ℓ.

$kQ := kQ_0 \oplus kQ_1 \oplus \cdots \oplus kQ_\ell \oplus \cdots$ as a vector space.

$$\overset{\alpha_1 \cdots \alpha_n}{\left(i \xleftarrow{\alpha_1} \bullet \leftarrow \cdots - \bullet \xleftarrow{\alpha_n} j \right)} \times \overset{\color{red}{e_{i'}}}{\overset{\beta_1 \cdots \beta_n}{\left(i' \xleftarrow{\beta_1} \bullet \leftarrow \cdots - \bullet \xleftarrow{\beta_m} j' \right)}}$$

$$:= \begin{cases} \overset{\color{red}{\alpha_1 \cdots \alpha_n}}{\alpha_1 \cdots \alpha_n} \beta_1 \cdots \beta_m & (j = i') \\ 0 & (j \neq i') \end{cases}$$

$$e_i \times e_j := \begin{cases} e_i & (i = j) \\ 0 & (i \neq j) \end{cases}$$

Ex

- $Q = 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$

$$kQ = \begin{array}{cl} ke_1 \oplus ke_2 \oplus ke_3 &]\, kQ_0 \\ \oplus & \\ k\alpha \oplus k\beta &]\, kQ_1 \\ \oplus & \\ k\alpha\beta &]\, kQ_2 \end{array}$$

$kQ_{\geq 3} = 0$

$$\simeq T_3(k) = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

- $Q = 1 \circlearrowleft \alpha$

$$kQ = \begin{array}{c} ke_1 \\ \oplus \\ k\alpha \\ \oplus \\ k\alpha^2 \\ \oplus \\ \vdots \end{array} \simeq k[x]$$

Rem Path algebras are quadratic ring.

∵) $R = 0$.

※ $\underset{i \leftarrow j}{\alpha}$, $\underset{i' \leftarrow j'}{\beta}$ $j \neq i'$

$$T_{kQ_0} kQ_1 \ni \alpha \underset{kQ_0}{\otimes} \beta = \alpha e_j \underset{kQ_0}{\otimes} \beta = \alpha \underset{kQ_0}{\otimes} e_j \beta = 0$$

Ex. of a quadratic ring

- A quadratic monomial algebra A

  :⇔ a finite dimensional algebra $A = kQ/I$ given with relation

  $I \ni a = a_1 + \cdots + a_n \Rightarrow a_i \in I$

  where I is two-sided homogeneous ideal generated by a set of paths of length two in Q.

- $Q: 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta$ $\qquad A = kQ/(\beta^2)$

$$\begin{array}{ccc} & A & \\ \vdots & & \vdots \\ A_{-1} & = & 0 \\ A_0 & = & ke_1 \oplus ke_2 \\ A_1 & = & k\alpha \oplus k\beta \\ A_2 & = & k\beta\alpha \\ A_3 & = & 0 \\ \vdots & & \vdots \end{array}$$

∴ A is a quadratic monomial algebra.

Def 13 Let $A = T_{A_0}A_1/(R)$: left locally finite quadratic ring over A_0.

$$A^! := T_{A_0}(A_1^*)/(R^\perp) \quad \text{is the } \underline{\text{left quadratic dual}} \text{ of } A$$

$$\left({}^!A := T_{A_0}({}^*A_1)/({}^\perp R) \right) \quad (\text{right} \quad \text{''} \quad)$$

with $R^\perp \subset A_1^* \otimes_{A_0} A_1^* = (A_1 \otimes_{A_0} A_1)^*$

Rem

- $A^!$ $({}^!A)$ is the right (left) finite quadratic ring over k.
- ${}^!(A^!) = A$ and $({}^!A)^! = A$

Ex

$$A := k\left(1 \overset{\alpha}{\underset{\gamma}{\leftrightarrows}} 2 \overset{\beta}{\underset{\delta}{\leftrightarrows}} 3 \right) \Big/ \left(\begin{matrix} \alpha\gamma, \\ \gamma\alpha + \beta\delta \end{matrix} \right)$$

quadratic monomialではない!

$$A \cong T_{A_0}A_1/(R) \qquad \text{where} \qquad \begin{aligned} A_0 &= kQ_0 \\ A_1 &= kQ_1 \\ R &= k\,\alpha \otimes_{A_0} \gamma \oplus k\left((\gamma \otimes_{A_0} \alpha) + (\beta \otimes_{A_0} \delta) \right) \end{aligned}$$

$\therefore$ A is a left locally finite quadratic ring.

- Calculate $R^\perp = \{ f \in (A_1 \otimes_{A_0} A_1)^* \mid f(R) = 0 \}$

Prop $\dim R^\perp + \dim R = \dim A_1 \otimes_{A_0} A_1$

$\therefore \dim R^\perp = 6 - 2 = 4$ つながった2つのarrowのペアの個数 $\rightarrow \bullet \rightarrow$

$\otimes = \otimes_{A_0}$

$$\begin{aligned} \therefore R^\perp &= k\left((\gamma \otimes \alpha)^* - (\beta \otimes \delta)^* \right) + k(\alpha \otimes \beta)^* + k(\delta \otimes \gamma)^* + k(\delta \otimes \beta)^* \\ &= k(\alpha^* \otimes \gamma^* - \delta^* \otimes \beta^*) + k(\beta^* \otimes \alpha^*) + k(\gamma^* \otimes \delta^*) + k(\beta^* \otimes \delta^*) \end{aligned}$$

$$A^! = k\left(1 \overset{\alpha^*}{\underset{\gamma^*}{\rightleftarrows}} 2 \overset{\beta^*}{\underset{\delta^*}{\rightleftarrows}} 3 \right) \Big/ \left(\begin{matrix} \alpha^*\gamma^* - \delta^*\beta^*, \\ \beta^*\alpha^*, \gamma^*\delta^*, \\ \beta^*\delta^* \end{matrix} \right)$$

Rem A = Auslander alg. of $k[x]/(x^3)$

$$A^! = (\text{Brauer tree alg } \circ\!-\!\circ\!-\!\circ\!-\!\circ) / \operatorname{soc} P_3$$

§5. Symmetric v.s. exterior algebras 1

Thm 14 For $V \in k\text{-mod}$,

$$S(V)^! = \Lambda(V^*).$$

proof)

$$S(V)^! = T^k(V^*)/(R^\perp)$$

$$R^\perp = \{ f \in (V\otimes V)^* \mid f(v\otimes w - w\otimes v) = 0 \quad \text{for } v, w \in V \}$$

$$\Lambda(V^*) = T_k(V)/(r) \qquad \text{where } r = \{ v^*\otimes v^* \in V^*\otimes V^* \mid v \in V \}$$
$$= \{ (v\otimes v)^* \in (V\otimes V)^* \mid v \in V \}$$

- $(R^\perp) = (\{ (v\otimes w)^* + (w\otimes v)^* \mid v, w \in V \})$

  ∵) $\supset$) $((x\otimes y)^* + (y\otimes x)^*)(v\otimes w - w\otimes v) = \begin{cases} 1 - 1 = 0 & (x = v \text{ and } y = w) \\ 0 & \text{otherwise} \end{cases}$

  $\subset$) 略.

- $(R^\perp) = (\{ (v\otimes v)^* \in (V\otimes V)^* \mid v \in V \})$

  ∵) $\subset$) Consider $((v+w)\otimes(v+w))^*$

  $\supset$) $(x\otimes x)^*(v\otimes w - w\otimes v)$
  $= \begin{cases} 0 & v = w = x \\ 0 & \text{otherwise.} \end{cases}$

$$\therefore\ S(V)^! = \Lambda(V^*)$$

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