Koszulity in quasi-hereditary algebras

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§1. Quasi-hereditary algebras

§2. Standard Koszul algebras.

§3. Balanced algebras.

Notations

$\mathbb{K}$: alg. closed field.

$A = \bigoplus_{i\geq 0} A_i$: basic fin-dim. pos-gr. $\mathbb{K}$-alg

$e_1, \ldots, e_n$: pair. orth. prim. idemp.

$$\left(A_0 = e_1\mathbb{K} \oplus \cdots \oplus e_n\mathbb{K}, \quad \operatorname{rad} A = \bigoplus_{i\geq 1} A_i \right)$$

$\Lambda = \{1 < \cdots < n\}$. $\quad \Lambda^{op} = \{1 > \cdots > n\}$.

$\lambda \in \Lambda$

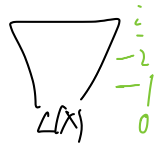
$P(\lambda) = Ae_\lambda$: gr. indec. proj A-module.

$L(\lambda) = P(\lambda)/\operatorname{rad} P(\lambda)$: simple A-mod.

$I(\lambda)$: inj. envelope of $L(\lambda)$.

$$\begin{cases} P(\lambda)_0 = P(\lambda)/\operatorname{rad} P(\lambda) \cong L(\lambda) \\ L(\lambda)_0 = L(\lambda) \\ I(\lambda)_0 = \operatorname{soc} I(\lambda) \cong L(\lambda) \end{cases}$$

$P(\lambda) =$ 

$I(\lambda) =$ 

grading shift $\langle - \rangle$, $\quad (M\langle i \rangle)_j := M_{i+j}$.

§1. Quasi-hereditary algebras

(Lie theory) (ring theory)

Highest weight category $\simeq$ $\underline{\underline{A}}$-mod

[Cline-Parshall-Scott '88] q.h.a.

- q.h.a. BGG cat. $\mathcal{O}$ alg. gr.

standard module $\sim$ Verma module $\sim$ Weyl module

- q.h.a. $\Leftarrow$ hereditary alg, Auslander alg

Def 1.1.

(i) The <u>standard module</u> $\Delta(\lambda)$, $\lambda \in \Lambda$

$$\Delta(\lambda) := P(\lambda) \Big/ \sum_{\star} \operatorname{Im} f$$

$\star$: $f : P(\mu)\langle j \rangle \to P(\lambda)$, $\mu > \lambda$, $j \in \mathbb{Z}$.

(or the max factor of $P(\lambda)$

s.t. $[\Delta(\lambda) : L(\mu)\langle j \rangle] \neq 0$ implies $\mu \leq \lambda$)

In particular, $\Delta(n) = P(n)$.

(ii) $\mathcal{F}(\Delta) \overset{\text{full}}{\subset} A\text{-gr}$

$M \in \mathcal{F}(\Delta)$ (M has a Δ-filtration).

$\overset{\text{def}}{\iff}$ $0 = M_m \subset \cdots \subset M_1 \subset M_0 = M$

s.t. $1 \leq {}^{\forall} k \leq m$, ${}^{\exists}\lambda \in \Lambda$, ${}^{\exists} j \in \mathbb{Z}$

$0 \to M_k \to M_{k-1} \to \Delta(\lambda)\langle j \rangle \to 0$.

$[M:\Delta(\lambda)\langle j\rangle]$: the number of $\Delta(\lambda)\langle j\rangle$ in a Δ-filt. of M.

(unique!)

(iii) A: <u>quasi-hereditary algebra</u> (q.h.a.)

$\overset{\text{def}}{\iff}$ ① $\lambda\in\Lambda$, $\mathrm{End}(\Delta(\lambda))\cong\mathbb{K}$

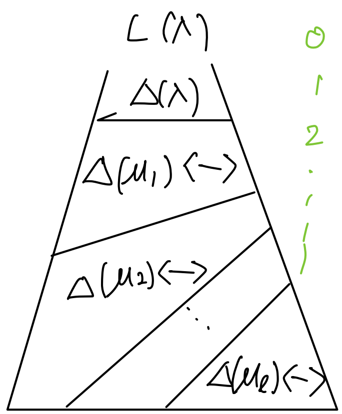
② ${}_AA\in\mathcal{F}(\Delta)$.

<u>Rmk</u> A^{op} is also a q.h.a. w.r.t. Λ

$$\nabla_A(\lambda) := \Delta_{A^{op}}(\lambda)^*.$$

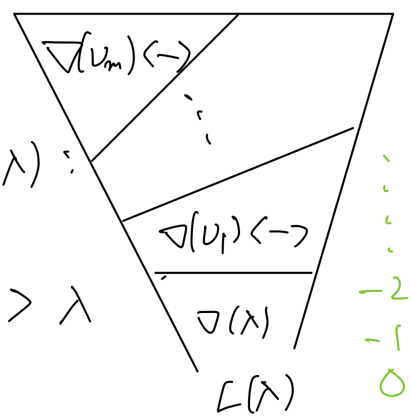
$P(\lambda)$:

$\mu_i > \lambda$



$I(\lambda)$:

$\nu_j > \lambda$



Ex 1.2

(i) A: hereditary alg. (w.r.t. any ordering).

$$Q: \overset{\lambda}{\circ} \longrightarrow \overset{\mu}{\circ} \longrightarrow \overset{\nu}{\circ}, \quad A = \mathbb{K}Q.$$

$$P(\lambda): \begin{matrix}\lambda\\ \mu\\ \nu\end{matrix} \qquad P(\mu): \begin{matrix}\mu\\ \nu\end{matrix} \qquad P(\nu): \nu$$

$\rightsquigarrow$ A: q.h.a. (6 cases)

we will consider the case $\mu < \lambda < \nu$

i.e. $A = \mathbb{K}(\overset{2}{\circ} \longrightarrow \overset{1}{\circ} \longrightarrow \overset{3}{\circ})$

(ii). Q: $\overset{1}{\circ} \underset{b_1}{\overset{a_1}{\rightleftarrows}} \overset{2}{\circ} \underset{b_2}{\overset{a_2}{\rightleftarrows}} \overset{3}{\circ} \cdots \underset{b_{n-1}}{\overset{a_{n-1}}{\rightleftarrows}} \overset{n}{\circ}$

$$I = (\overset{\rightarrow\rightarrow}{a_{i+1}a_i},\ \overset{\leftarrow\leftarrow}{b_i b_{i+1}},\ \overset{\circlearrowleft\, -\, \circlearrowright}{a_i b_i - b_{i+1}a_{i+1}},\ \overset{\circlearrowleft n}{b_{n-1}a_{n-1}})$$

$$A_{(n)} = \mathbb{K}Q/I \qquad \begin{matrix}\deg a_i = 1\\ \deg b_i = 1.\end{matrix}$$

$$\begin{array}{c} \Delta(1) \\ P(1) = \begin{array}{c} \boxed{1} \\ 2 \\ \boxed{1} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \\ \shortparallel \\ I(1)\langle -2\rangle \quad \nabla(1) \end{array} \qquad \begin{array}{c} \Delta(\lambda) \\ P(\lambda) = \begin{array}{ccc} & \lambda & \\ \lambda-1 & & \lambda+1 \\ & \lambda & \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \\ \shortparallel \\ I(\lambda)\langle -2\rangle \quad \nabla(\lambda) \end{array} \qquad \begin{array}{c} \Delta(n) \\ P(n) = \boxed{\begin{array}{c} n \\ n-1 \end{array}} \begin{array}{c} 0 \\ 1 \end{array} \\ \\ I(n) = \boxed{\begin{array}{c} n-1 \\ n \end{array}} \begin{array}{c} -1 \\ 0 \end{array} \\ \nabla(n) \end{array}$$

$$(2 \leq \lambda \leq n-1)$$

Δ-filt. $\qquad\qquad$ ∇-filt.

$$0 \subset \Delta(2)\langle -1\rangle \subset P(1) \qquad I(1) \longrightarrow \nabla(2)\langle 1\rangle \to 0$$

$$0 \subset \Delta(\lambda+1)\langle -1\rangle \subset P(\lambda) \qquad I(\lambda) \longrightarrow \nabla(\lambda+1)\langle 1\rangle \to 0$$

$$0 \subset P(n) \qquad I(n) \longrightarrow 0$$

This alg. is the Koszul dual of Auslander of $\mathbb{k}[x]/(x^n)$.

Prop 1.3 [Dlab-Ringel '91].

(i) $\mathrm{gl.dim}\, A \leq 2n-2$ ← Ex 1.2 (ii).

So $E(A)$ is fin. dim.

(ii) (Brauer-Humphreys reciprocity).

$$[P(\lambda) : \Delta(\mu)] = [\nabla(\mu) : L(\lambda)].$$

(iii) $$\mathrm{ext}^i_A(\Delta(\lambda), \nabla(\mu)\langle j\rangle) = \begin{cases} \mathbb{K} & (\lambda=\mu,\ i=j=0) \\ 0 & \text{otherwise.} \end{cases}$$

(iv) TFAE. (Not assume A is q.h.)

① ${}_A A \in \mathcal{F}(\Delta)$

② $\mathrm{Ext}^2(\Delta, \nabla) = 0$

③ $\mathcal{F}(\Delta) = \{ X \in A\text{-gr} \mid \mathrm{Ext}^1(X, \nabla) = 0 \}$.

(v) There is the <u>(characteristic) Tilting module.</u>

$$T = \bigoplus_{\lambda \in \Lambda} T(\lambda) \quad \text{in} \quad A\text{-mod}$$

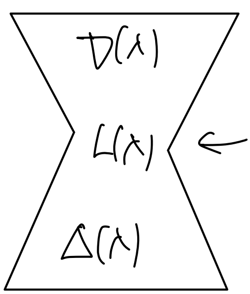
with exact sequences

$$0 \to \Delta(\lambda) \to T(\lambda) \to X \to 0$$
$$0 \to Y \to T(\lambda) \to \nabla(\lambda) \to 0$$

for $X \in \mathcal{F}(\Delta_{>\lambda})$, $Y \in \mathcal{F}(\nabla_{>\lambda})$.

s.t. $\operatorname{add} T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

☆ $[T(\lambda) : L(\lambda)]_{A\text{-mod}} = 1 \rightsquigarrow T(\lambda)_0 \supset L(\lambda)$.

$T(\lambda) =$ 

← unique

Def 1.4

$$R(A) := \mathrm{End}_A(T)^{op}$$

$$= \left(\bigoplus_{i \in \mathbb{Z}} \hom_A(T, T\langle i \rangle) \right)^{op} : \text{Ringel dual.}$$

(not necessarily pos-gr., see Thm 3.1)

Rmk

$R(A)$ is a g.h.a. w.r.t. Δ^{op}.

The first Ringel duality functor

$$F := \bigoplus_{i \in \mathbb{Z}} R\hom_A(T\langle i \rangle, -) : D^b(A) \xrightarrow{\sim} D^b(R(A))$$

maps

$$T_A \longmapsto P_{R(A)}$$

$$\nabla_A \longmapsto \Delta_{R(A)}$$

$$I_A \longmapsto T_{R(A)}.$$

The second Ringel duality functor

$$G = \bigoplus_{i \in \mathbb{Z}} \mathbb{R}\mathrm{hom}_A(-, T\langle i \rangle)^* : \mathcal{D}^b(A) \xrightarrow{\sim} \mathcal{D}^b(R(A))$$

maps
$$T_A \to I_{R(A)}$$
$$\Delta_A \to \nabla_{R(A)}$$
$$P_A \to T_{R(A)}.$$

Let $N_A = A^* \otimes^{\mathbb{L}}_A -$

be the Nakayama functor

Then
- $N_A(P_A(\lambda)) \cong I_A(\lambda)$
- $G = N_{R(A)} \circ F$

Ex 1.5

(i) $A = K\left(\overset{2}{\circ} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{3}{\circ} \right)$

$$P_A = \begin{array}{c}\boxed{1}\\3\end{array} \oplus \boxed{\begin{array}{c}2\\1\end{array}}\begin{array}{c}\\ \\3\end{array} \oplus \boxed{3}$$

$$I_A = \begin{array}{c}2\\\boxed{1}\end{array} \oplus \boxed{2} \oplus \boxed{\begin{array}{c}2\\1\\3\end{array}}$$

$$T_A = 1 \;\; 0 \;\oplus\; \begin{array}{cc}2 & 0\\1 & 1\end{array} \;\oplus\; \begin{array}{cc}2 & -2\\1 & -1\\3 & 0\end{array} \qquad \text{deg.}$$

$R(A) = K\left(\overset{1}{\cdot} \xleftarrow{1} \overset{2}{\cdot} \xrightarrow{2} \overset{3}{\cdot} \right)$: q.h.a.

(ii) $A_{(n)} = K\left(\overset{1}{\cdot} \rightleftarrows \overset{2}{\cdot} \cdots \rightleftarrows \overset{n}{\cdot} \right) \cancel{/ \langle \to\to, \leftarrow\leftarrow, \ldots \rangle}$

$T(1) : 1 \;\; 0$, $T(2) : \begin{array}{cc}1 & -1\\2 & 0\\1 & 1\end{array}$, $T(\lambda) : \begin{array}{ccc} & \lambda-1 & -1\\ \lambda-2 & \lambda & 0\\ & \lambda-1 & 1\end{array}$

$(\lambda \geq 3)$

$R(A_{(n)}) \cong A_{(n)}$

§ 2. Standard Koszul algebras.

Q. The Koszul dual of Koszul q.h.a. is also q.h.a.?

NO!!

Ex.

$$A = \mathbb{K}\left(\overset{2}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{3}{\bullet}\right)$$

$$E(A) = \mathbb{K}\left(\overset{3}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}\right) \Big/ \langle \longrightarrow\!\!\!\rightarrow \rangle$$

$$P_{E(A)} = \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}} \oplus \boxed{2} \oplus \begin{smallmatrix}\boxed{3}\\1\end{smallmatrix}$$

$\Longrightarrow$ $E(A)$ is NOT q.h.

A complex

$$P^\bullet : \cdots \longrightarrow P^{i-1} \xrightarrow{d^{i-1}} P^i \xrightarrow{d^i} P^{i+1} \longrightarrow \cdots$$

of proj. (resp. inj., resp. tilt.)

A-modules is called <u>linear</u>

if $\forall i \in \mathbb{Z}$, $P^i \in \operatorname{add}\{A\langle i\rangle\}$

$\uparrow$

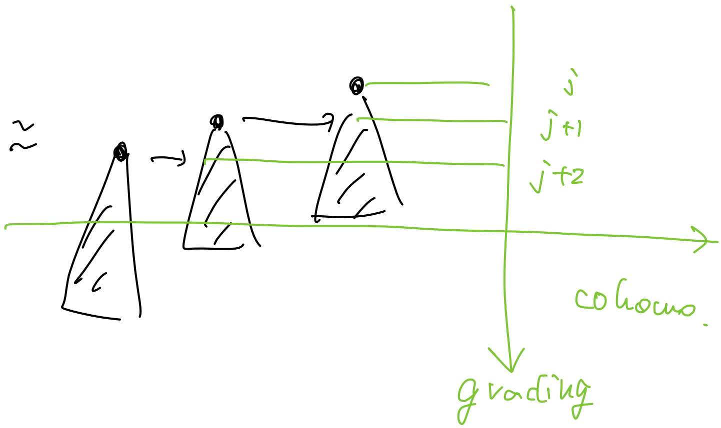
A^*, T

$$\mathcal{LP}, \mathcal{LI}, \mathcal{LT} \overset{\text{full}}{\subset} \mathcal{D}^b(A).$$

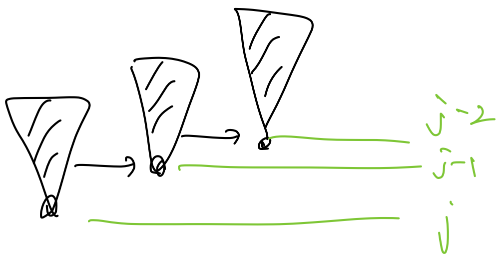
cat's of linear complexes of

proj. / inj. / tilt. A-modules, resp.

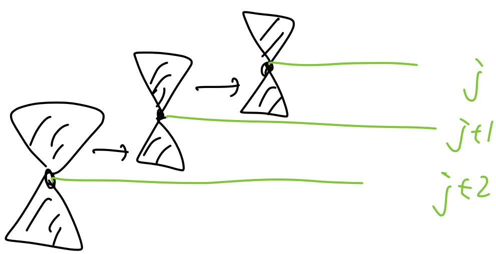
$\mathcal{I}p \ni P^\bullet \simeq$



$\mathcal{I}d \ni I^\bullet \simeq$



$\mathcal{L}\mathcal{T} \ni T^\bullet \simeq$



Def A pos-gr. q.h.a. is called

Koszul

$$\mathcal{P}(L)^{\circ} \in \mathcal{I}\mathcal{P}. \quad (L \in \mathcal{I}\mathcal{P})$$

Standard Koszul

$$\mathcal{P}(\Delta)^{\circ} \in \mathcal{I}\mathcal{P} \qquad \left(\begin{array}{l} \Delta \in \mathcal{I}\mathcal{P} \\ \nabla \in \mathcal{I}\mathcal{J} \end{array} \right)$$

$$\mathcal{J}(\nabla)^{\circ} \in \mathcal{I}\mathcal{J}$$

$$E(A) := \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(L, L)^{op} \; : \text{Koszul dual.}$$

The Koszul duality functor

$$K_A = \bigoplus_{i \in \mathbb{Z}} \mathbb{R}\hom_A(L\langle i\rangle[-i], -) : \mathcal{D}^b(A) \xrightarrow{\sim} \mathcal{D}^b(E(A))$$

$$K \circ \langle 1 \rangle_A = \langle -1 \rangle_{E(A)} \circ [1] \circ K$$

Rmk

We have the following comm. diag's.

$$\begin{tikzcd}
 & \mathcal{L}\mathcal{d} \\
\mathcal{L}\mathcal{J} \arrow[ur, "G", "\sim"'] \arrow[dr, "\sim", "F"'] \arrow[r, phantom, "\circlearrowleft"] & \\
 & \mathcal{I}\mathcal{P} \arrow[uu, "\wr"', "N_{R(A)}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{L}\mathcal{d} \arrow[dr, "K_A", "\sim"'] \arrow[r, phantom, "\circlearrowleft"] & \\
 & E(A)\text{-}gr \\
\mathcal{I}\mathcal{P} \arrow[uu, "\wr", "N_A"'] \arrow[ur, "\sim", "K^{-1}_{E(A)}"'] &
\end{tikzcd}$$

,

(gr. ver. of)

Main theorem [Ágoston–Dlab–Lukács '03]

(i) Standard Koszul q.h.a.

$\Longrightarrow$ Koszul. ($\not\Longleftarrow$ Ex. 2.3(i))

(ii) A: standard Koszul q.h.a.

$\Longrightarrow E(A)$: standard Koszul q.h.a.

Prop. 2.1

Let A be standard Koszul.

Then so is $e_{\geq\lambda} A e_{\geq\lambda}$ for all $\lambda \in \Lambda$,

where $e_{\geq\lambda} = \sum_{\mu \geq \lambda} e_\mu$

Thm 2.2

Standard Koszul alg is Koszul.

proof) Induction on n $(= \#\Lambda)$

$n = 1 \Rightarrow L(1) = \Delta(1) \in \mathcal{I}\mathcal{P}$ ok.

Let $n \geq 2$.

$\boxed{\text{Aim}}$

$\hom_{D^b(A)}(L, L\langle -j \rangle [i]) = 0$ if $i \neq j$

$L(1) = \Delta(1) = \nabla(1) \in \mathcal{IP} \cap \mathcal{I}\ell.$

So for $i \neq j$,

$$\hom_{\mathcal{D}^b(A)}(L(1), L\langle -j\rangle[i]) = \hom_{\mathcal{D}^b(A)}(L, L(1)\langle -j\rangle[i]) = 0.$$

Put $L' = \bigoplus_{\lambda \geq 2} L(\lambda)$

Show $i \neq j \Rightarrow \hom_{\mathcal{D}^b(A)}(L', L'\langle -j\rangle[i]) = 0.$

$$0 \to A\varepsilon A \to A \to A/A\varepsilon A \to 0$$

$$\Big\downarrow \mathbb{R}\hom_{\mathcal{D}^b(A)}(- \otimes^{\mathbb{L}}_A L', L'\langle -j\rangle)$$

$$\to \hom(A\varepsilon A \otimes^{\mathbb{L}}_A L', L'\langle -j\rangle[i]) \to \hom(L', L'\langle -j\rangle[i])$$

$$\to \hom(A/A\varepsilon A \otimes L', L'\langle -j\rangle[i]) \to$$

$\uparrow$ $A/A\varepsilon A \otimes^{\mathbb{L}}_A L' \simeq \bigoplus_{i \geq 1} L(1)\langle -k\rangle[k].$

<u>Fact</u> A : q.h.a.

$$\Longrightarrow A\varepsilon A \cong A\varepsilon \otimes^{L}_{\varepsilon A\varepsilon} \varepsilon A.$$

$$\mathrm{ext}^{i}_{A}(A\varepsilon A \otimes^{L}_{A} L', L'\langle -j\rangle[i]) \cong \mathrm{ext}^{i}_{\varepsilon A\varepsilon}(\varepsilon L', \varepsilon L'\langle -j\rangle)$$

$$= 0 \quad \text{if } i \neq j.$$

by assumption "$\varepsilon A\varepsilon$: Koszul." $\square$

Ex 2.3

$$A_{(n)} = \Bbbk\left(\overset{1}{\bullet} \rightleftarrows \overset{2}{\bullet} \cdots\cdots \rightleftarrows \overset{n}{\bullet}\right) \Big/ \langle \to\to,\ \leftarrow\leftarrow,\ \circlearrowleft - \circlearrowright,\ \circlearrowleft_n \rangle$$

For any $\lambda \in \Lambda$,

$$0 \to \Delta(\lambda+1)\langle -1 \rangle \to P(\lambda) \to \Delta(\lambda) \to 0$$

$\Delta(n+1) = 0.$

$$0 \to \nabla(\lambda) \to I(\lambda) \to \nabla(\lambda+1)\langle 1 \rangle \to 0$$

$\nabla(n+1) = 0.$

A is standard Koszul.

$^{\forall}\lambda \in \Lambda$,

$$e_{\geq\lambda} A_{(n)} e_{\geq\lambda} \cong A_{(n-\lambda+1)}$$

are standard Koszul.

Prop 2.4

Let A be a standard Koszul q.h.a.

Then $K_A(\nabla_A(\lambda)) \cong \Delta_{E(A)}(\lambda)$

$K_A(N_A(\Delta(\lambda))) \cong \nabla_{E(A)}(\lambda)$

w.r.t. $\leq^{op}$.

proof) Show $K_A(\nabla_A(\lambda)) \cong \Delta_{E(A)}(\lambda)$.

check that

① $\operatorname{top} K(\nabla_A(\lambda)) \cong L_{E(A)}(\lambda)$

② $\mu > \lambda \Rightarrow \operatorname{ext}^1_{E(A)}(K(\nabla_A(\lambda)), L_E(\mu)\langle j' \rangle) = 0$

③ $[K(\nabla_A(\lambda)) : L_{E(A)}(\mu)\langle j \rangle] = 0 \Rightarrow \mu \geq \lambda$.

(see [DR].)

①, ②

$$\hom_{D^b_{gr}(E(A))}\left(K(\nabla_A(\lambda)),\ L_E(\mu)\langle -j\rangle_E [i]\right)$$

$$\overset{K^{-1}}{\cong} \hom_{D^b(A)}\left(\nabla_A(\lambda),\ I_A(\mu)\langle j\rangle_A [i-j]\right)$$

$$\cong \mathrm{ext}^{i-j}_A\left(\nabla_A(\lambda),\ I_A(\mu)\langle j\rangle\right)$$

This is iso to. $\mathbb{K}$ if $i=j=0$, $\lambda=\mu$ ①

0 if $i=1$, $\underline{\lambda<\mu}$ ②

∴ $j=1$

③

radical series (upper → lower) of $K(\nabla_A(\lambda))$ in $E(A)$-gr	$\longleftrightarrow$	linear inj coresol. of $\nabla_A(\lambda)$ in A-gr.

For simplicity, inj dim $\nabla_A(\lambda) = 1$.

$$0 \to \nabla_A(\lambda) \to I^0 \to I^1\langle 1\rangle \to 0$$

$I^1 \in \mathrm{add}\{ I(\mu) \mid \mu > \lambda \}$. in A-gr.

$$I^1\langle 1\rangle_A[-1] \to \nabla_A(\lambda) \to I^0 \to \quad \text{in } \mathcal{D}^b(A)$$

$$K \Big\downarrow$$

$$0 \to K(I^1)\langle -1\rangle_E \to K(\nabla_A(\lambda)) \to K(I^0) \to 0 \quad \text{in } E(A)\text{-gr}.$$

$K(I^1)\langle -1\rangle = L_1\langle -1\rangle$, $\quad K(I^0) = K(I_A(\lambda)) \cong L_E(\lambda)$

$L_1 \in \mathrm{add}\{ L(\mu) \mid \mu > \lambda \}$.

$$\therefore K(\nabla(\lambda)) = \begin{matrix} L_E(\lambda) & 0 \\ L_1\langle -1\rangle & 1 \end{matrix}$$

In general,

$$0 \to \nabla(\lambda) \to I^0 \to \cdots \to I^m\langle m\rangle \to 0$$

$$\rightsquigarrow \quad K(\nabla(\lambda)) = \begin{matrix} L_E(\lambda) \\ \vdots \\ L_m\langle -m\rangle. \end{matrix}$$

$\square$

Thm 2.5

Let A be a standard Koszul g.h.a.

Then $E(A)$ is a g.h.a. w.r.t. Δ^{op}

proof) Check
$$\begin{cases} ① \ E(A) \in \mathcal{F}(\Delta_E). \\ ② \ \mathrm{End}(\Delta_E(\lambda)) \cong \Bbbk. \end{cases}$$

Use Prop 1.3 (iv)

$$E(A) \in \mathcal{F}(\Delta) \iff \mathrm{Ext}^2(\Delta, \nabla) = 0.$$

$$\mathrm{ext}^2_{E(A)}(K(\nabla_A), K \circ N_A(\Delta_A)\langle j \rangle_E)$$

$$\overset{K^{-1}}{\cong} \hom_{D(A)}(\nabla_A, N_A(\Delta_A)\langle -j \rangle_A [2+j])$$

$$\cong \hom_{D(A)}(\Delta_A\langle -j \rangle[2+j], \nabla_A)^*$$

$$\cong \mathrm{ext}^{-2-j}(\Delta_A, \nabla_A\langle j \rangle)^* = 0.$$

② $\mathrm{hom}_{E(A)}\left(K(\nabla_A(\lambda)),\ K(\nabla_A(\lambda))\langle j\rangle_E \right)$

$$\overset{KT}{\cong} \bigoplus_{j\in\mathbb{Z}} \mathrm{ext}^j\left(\nabla_A(\lambda),\ \nabla_A(\lambda)\langle -j\rangle_A\right).$$

$$\mathrm{ext}^j\left(\nabla_A(\lambda),\ \nabla_A(\lambda)\langle -j\rangle\right)$$

$$\cong \begin{cases} \mathbb{K} & (j=0) \\ 0 & (j\neq 0) \end{cases}$$

$j<0$ $\mathrm{ext}^j = 0$.

$j>0$ $(\underline{\qquad\qquad})$

$d\left(\nabla(\lambda)\langle -j\rangle\right)^j \in \mathrm{add}\{I(\mu) \mid \mu>\lambda\}$.

$\square$

Thm 2.6.

Let A be a standard Koszul q.h.a.
Then $E(A)$ is also standard Koszul.

Proof)

$$\begin{tikzcd} & \mathcal{LP}_E \\ A\text{-gr} \arrow[ur, "K", "\sim"'] \arrow[dr, "\sim", "K_E^{-1}"'] & \\ & \mathcal{LI}_E. \end{tikzcd}$$

Ex. 2.7 $\quad A(i) \cong E(A(i))$ for $i = 1, 2$.

Let $n = 3$

$$E(A_{(3)}) = \mathbb{K}\left(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 3 \right) \Big/ \left\langle \begin{matrix} \alpha\beta = \delta\gamma \\ \beta\alpha \end{matrix} \right\rangle$$

$$P_E = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \;\begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \;\oplus\; \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \\ & 3 & \end{matrix} \;\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \;\oplus\; \begin{matrix} & 3 & \\ & 2 & \\ 1 & & 3 \\ & 2 & \\ & 3 & \end{matrix} \;\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$I_E = \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \;\oplus\; \begin{matrix} & 3 & \\ & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix} \;\oplus\; \begin{matrix} & 3 & \\ & 2 & \\ 1 & & 3 \\ & 2 & \\ & 3 & \end{matrix}$$

$$P(1) \xrightarrow{\sim} \Delta(1)$$

$$P(3)\langle -1 \rangle \longrightarrow P(2) \xrightarrow{\sim} \Delta(2)$$

$$P(2)\langle -1 \rangle \longrightarrow P(3) \xrightarrow{\sim} \Delta(3)$$

$$\nabla(1) \xrightarrow{\sim} I(1)$$

$$\nabla(2) \xrightarrow{\sim} I(2) \longrightarrow I(3)\langle 1 \rangle$$

$$\nabla(3) \xrightarrow{\sim} I(3) \longrightarrow I(2)\langle 1 \rangle$$

$$\Delta \in \mathcal{I}\mathcal{P}, \quad \nabla \in \mathcal{I}\mathcal{I}.$$

§3. Balanced algebras.

If A is pos-gr., so is $R(A)$?

See [Maz] Ex

Def

A pos-gr. q.h.a. is called <u>balanced</u>

$$0 \to \Delta(\lambda) \longrightarrow T(\lambda) \to T^1 \to \cdots$$

(linear)

$$\cdots \to T_1 \to T(\lambda) \to \nabla(\lambda) \to 0$$

$\Delta(\lambda), \nabla(\lambda) \in \mathcal{L}\mathcal{T}$

Thm 3. ([Mazorchuk – Ovsienko] Theorem 7)

Let A be a pos-gr. q.h.a.

TFAE

(i) A is balanced

(ii) A is standard Koszul and $R(A)$ is pos-gr.

Main theorems. [Mazorchuk '10]

Let A be a pos-gr. balanced alg.

(i) $E(R(A)) \cong R(E(A))$

(ii) $R(A), E(A)$: balanced.

Let A be a pos-gr. balanced q.h-a.

Then $R(A)$ is standard Koszul.

$\mathcal{I}(\nabla_R)^{\circ} \cong G(\mathcal{T}(\Delta_A)^{\circ})$, $\mathcal{P}(\Delta_R)^{\circ} \cong F(\mathcal{T}(\nabla_A)^{\circ})$

Rmk

We have the following comm. diag.

$$\begin{tikzcd}
 & \mathcal{L}\mathcal{I}_{R(A)} \arrow[dr, hook, "K_R"] & \\
\mathcal{L}\mathcal{T}_A \arrow[ur, "G", "\sim"'] \arrow[dr, "\sim", "F"'] \arrow[r, phantom, "\circlearrowright"] & \arrow[u, "\wr", "N_R"'] \arrow[r, phantom, "\circlearrowright"] & E(R(A))\text{-gr}. \\
 & \mathcal{L}\mathcal{P}_{R(A)} \arrow[ur, "\sim", "K^{-1}_{E(R)}"'] &
\end{tikzcd}$$

Write $H := K_R \circ G = K^{-1}_{E(R)} \circ F$

$(H \circ \langle 1 \rangle_A = \langle -1 \rangle_{E(R(A))} \circ [1] \circ H)$

Prop 3.2

$L_A \in \mathcal{I}\mathcal{J}$

$\Rightarrow H(\Delta_A), H(\nabla_A), H(L_A) \in E(R|A))$-gr.

Prop 3.3

$\Delta_{E(R)} \cong H(\Delta_A)$

$\nabla_{E(R)} \cong H(\nabla_A)$ w.r.t $\leq$.

proof)

$$H: \begin{array}{ccccc} \Delta_A & \overset{G}{\longmapsto} & \nabla_R & \overset{K_R}{\longmapsto} & \Delta_{E(R)} \\ \nabla_A & \underset{F}{\longmapsto} & \Delta_R & \underset{K_{E(R)}^{-1}}{\longmapsto} & \nabla_{E(R)} \end{array}$$

Prop 3.4

$$T_{E(R)} \cong H(L_A) \quad \text{w.r.t. } \leq .$$

Proof) By Prop 1.3 (iii), (v),

$$\begin{cases} \cdot \operatorname{Ext}^1_{E(R(A))}(H(\Delta_A), H(L_A)) = 0 & \cdots (1) \\ \cdot \operatorname{Ext}^1_{E(R(A))}(H(L_A), H(\nabla_A)) = 0 . & \cdots (2) \end{cases}$$

(1) $\hom_{\mathcal{D}^b(E(R))}(H(\Delta_A), H(L_A)\langle j\rangle[1])$

$\overset{H^{-1}}{\cong} \hom_{\mathcal{D}^b(A)}(\Delta_A, L_A\langle -j\rangle[1+j])$

$\cong \operatorname{ext}^{1+j}_A(\Delta_A, L_A\langle -j\rangle)$

$= 0 \qquad \because A$: pos-gr.

Similarly show (2).

Thm 3.5 ([Maz] Cor 15)

Let A be a pos-gr. balanced alg.

$$R(E(A)) \simeq E(R(A))$$

Proof)

$$R(E(R(A))) \cong \mathrm{End}_{E(R(A))}(H(L_A))^{op} \quad \langle j \rangle$$

$$\cong \bigoplus_{i \geq 0} \mathrm{Ext}^i_A(L_A)^{op} \quad \langle -j \rangle [j]$$

$$\cong E(A) \quad \langle j \rangle$$

Ex 3.6

$$A_{(3)} = \mathbb{K}\left(\overset{1}{\cdot} \rightleftarrows \overset{2}{\cdot} \rightleftarrows \overset{3}{\cdot} \right) \Big/ \left\langle 1 \to\to 3,\ 1 \leftarrow\leftarrow 3,\ G_2 - P,\ G_3 \right\rangle$$

$R(A_{(3)}) \cong A_{(3)}$: balanced.

Indeed $\Delta(1) = \nabla(1) = T(1)$

$$\Delta(2) \hookrightarrow T(2) \longrightarrow T(1)\langle 1 \rangle$$

$$\Delta(3) \hookrightarrow T(3) \longrightarrow T(2)\langle 1 \rangle \longrightarrow T(1)\langle 2 \rangle$$

$$T(1)\langle -1 \rangle \longrightarrow T(2) \longrightarrow \nabla(2)$$

$$T(1)\langle -2 \rangle \longrightarrow T(2)\langle -1 \rangle \longrightarrow T(3) \longrightarrow \nabla(3)$$

$$E(A_{(3)}) = \mathbb{K}\left(\overset{1}{\circ} \rightleftarrows \overset{2}{\circ} \rightleftarrows \overset{3}{\circ} \right) \Big/ \left\langle \begin{matrix} \circlearrowleft \overset{2}{-} \circlearrowright \\ 1 \circlearrowright 2 \end{matrix} \right\rangle \qquad (3 < 2 < 1)$$

$$\Delta_E = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3$$

$$\nabla_E = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus 3$$

$$T_E = \begin{smallmatrix} & 3 & -2 \\ & 2 & -1 \\ 1 & 3 & 0 \\ & 2 & 1 \\ & 3 & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & -1 \\ 2 & 0 \\ 3 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & 0 \end{smallmatrix}$$

$$R\,E(A_{(3)}) = \mathbb{K}\left(\overset{1}{\bullet} \rightleftarrows \overset{2}{\bullet} \rightleftarrows \overset{3}{\bullet} \right) \Big/ \left\langle \begin{matrix} \circlearrowleft \overset{2}{-} \circlearrowright \\ \circlearrowright 3 \end{matrix} \right\rangle$$

Hence

$$R(E(A_3)) \cong E(A_{(3)}) \cong E\,R(A_{(3)})$$