THE EXISTENCE OF BALANCED NEIGHBORLY POLYNOMIALS

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Join work with Professor Satoshi Murai

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Outline

- Introduction and motivation
- Main results

Simplicial complex

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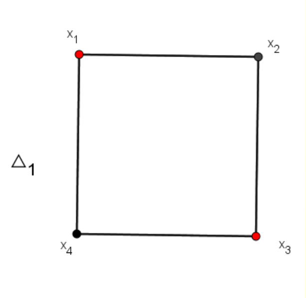
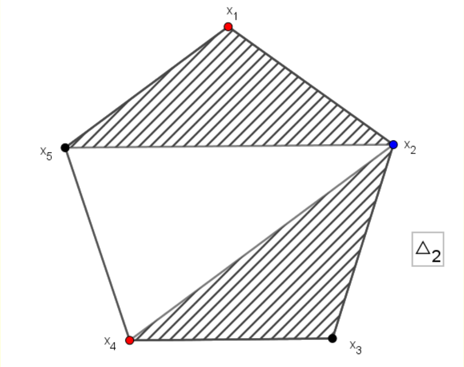
  $\dim\Delta = \max\{\dim F \mid F \in \Delta\}$: The *dimension of* Δ,
- Faces of dimension 0 are called *vertices* and faces of dimension 1 are called *edges.*

Example





$$\dim\Delta_1 = 1, \dim\Delta_2 = 2$$

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- $\dim K[\Delta] = \dim \Delta + 1$

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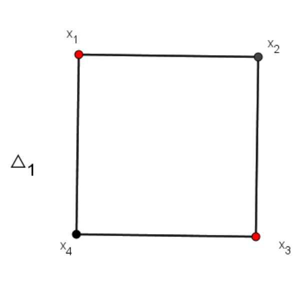

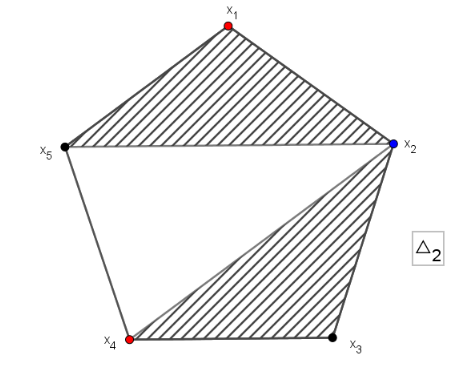

$$I_{\Delta_1} = (x_1x_3, x_2x_4), \dim K[\Delta_1] = 2$$

$$I_{\Delta_2} = (x_1x_3, x_1x_4, x_3x_5, x_2x_3x_4), \dim K[\Delta_2] = 3$$

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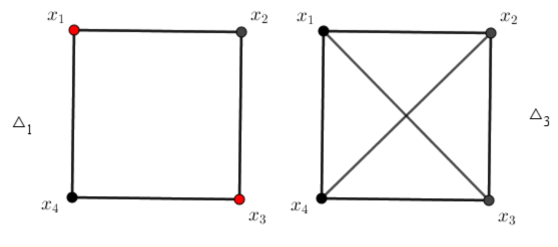

We have $\dim\Delta_1 = 1, \dim\Delta_3 = 1$.
Then Δ_1 is a balanced complex. Δ_3 is not a balanced complex.

Simplicial spheres

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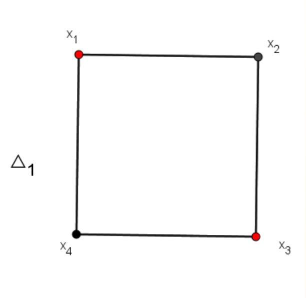

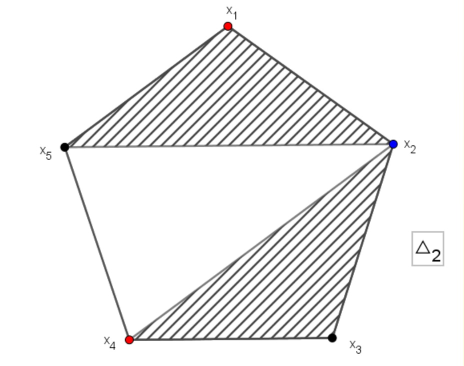

We have Δ_1 is simplicial sphere, but Δ_2 is not simplicial sphere

System of parameters of simplicial complex

- If $\dim K[\Delta] = d$ and a sequence Θ of linear forms such that

$$\dim_K K[\Delta]/(\Theta) < \infty$$

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- Set $V_k = \{v \in [n] \mid \kappa(v) = k\}$. Let $\theta_k = \sum_{v_k \in V_k} x_{v_k}$, for $k = 1, 2, \ldots, d$. Then $\Theta = \theta_1, \ldots, \theta_d$ is a l.s.o.p of $K[\Delta]$.

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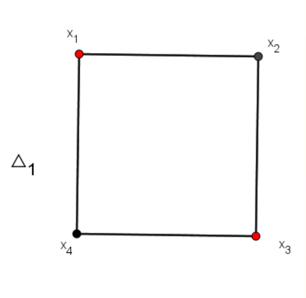

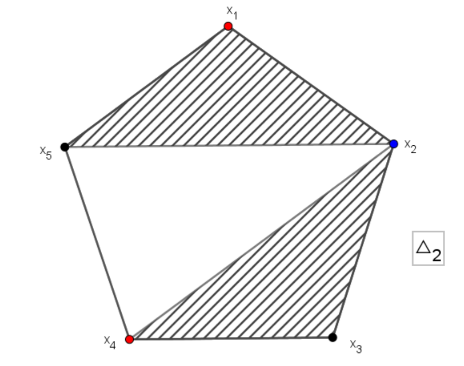

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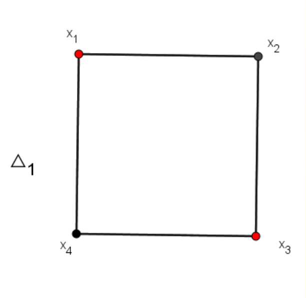

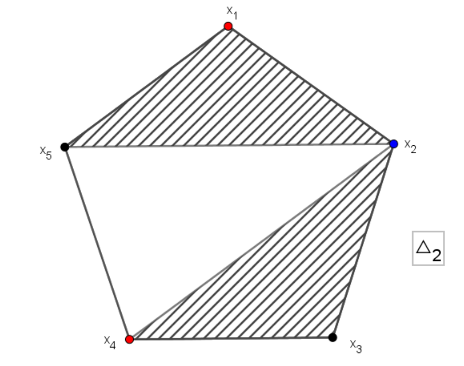

We have $\Theta_1 = (x_1 + x_3, x_2 + x_4) \Rightarrow \dim_K K[\Delta_1]/(\Theta_1) = 2.$
We have $\Theta_2 = (x_1 + x_4, x_3 + x_5, x_2) \Rightarrow \dim_K K[\Delta_2]/(\Theta_2) = 2.$

MOTIVATION

- [H. Zheng, 2020] A balanced simplicial sphere of dimension $d-1$ is called *neighborly of type* $(n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ if
$$\dim_K (K[\Delta]/(\Theta))_{e_S} = \prod_{k \in S} n_k,$$
for all $S \subset [d]$, $e_S = \sum_{i \in S} e_i \in \mathbb{Z}^d$, where $e_1, \ldots, e_d$ are the unit vectors of $\mathbb{Z}^d$.

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- [H. Zheng, 2020] Prove that: balanced neighborly spheres of type $(2,2,2,2)$ do not exist, but type $(3,3,3,3)$ exist.

H. Zheng, Ear decomposition and balanced neighborly simplicial manifolds, *The Electronic Journal of Cominatorics*, **27** (2020), P1.10.

Introduction

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- $f \in R$ is a *balanced polynomial* if $\deg(f) = (1, 1, \ldots, 1)$.
- For a balanced polynomial $f \in R$, let us consider the algebra of differential operators over R. Set

$$H(f, S) = \dim_K \{g(\partial_{ij}) f \mid g(x_{ij}) \in R_{\mathrm{e}_S}\},$$

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- A balanced polynomial f is *neighborly of type $(n_1, \ldots, n_d) \in \mathbb{Z}^d$* if $H(f, S) = \prod_{k \in S} n_k$, $\forall S \subseteq [d]$ with $|S| \leq \dfrac{d}{2}$.

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3. We proved that if balanced neighborly simplicial spheres of type $(n_1, n_2, \cdots, n_d)$ exist, then balanced neighborly polynomials of type $(n_1, n_2, \cdots, n_d)$ exist over any field K.

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4. Give an alternative proof for H. Zheng's result proving that balanced neighborly simplicial spheres of type $(2,2,2,2)$ do not exist.

Main results

In case d is even, we have the following Lemma.

Lemma 2

Let f be a balanced neighborly polynomial of type $(n_1, \cdots, n_d) \in \mathbb{Z}^d$. If d is even then $n_1 = n_2 = \cdots = n_d$.

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- From now, we consider $d = 4, n_1 = n_2 = n_3 = n_4 = k$, i.e. $R = K[x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k, w_1, \ldots, w_k]$,

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- If $f \in R$ is a balanced polynomial, then

$$f = \sum_{(i_1,i_2,i_3,i_4) \in \{1,2,\ldots,k\}^4} a_{i_1,i_2,i_3,i_4} x_{i_1} y_{i_2} z_{i_3} w_{i_4},$$

where $a_{i_1,i_2,i_3,i_4} \in K$.

Main results

How?

Study about the existence of balanced neighborly polynomials of type (k, k, k, k) in the following cases:

- $k = 2$.
- k is odd.
- k is even and $k = 4m$.
- k is even and $k = 4m + 2$.

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Theorem 2

Let $k \in \mathbb{N}$ be odd and $f = \sum_{1 \leq i,j \leq k} x_i y_j z_{[j-i]_k} w_{[i+j]_k}$. Then f is a balanced neighborly polynomial of type (k,k,k,k).

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Example: $f = x_1y_1z_3w_2 + x_1y_2z_1w_3 + x_1y_3z_2w_1 + x_2y_1z_2w_3 + x_2y_2z_3w_1 + x_2y_3z_1w_2 + x_3y_1z_1w_1 + x_3y_2z_2w_2 + x_3y_3z_3w_3$

Theorem 3

From now on we assume that k is even. Let $\overline{x}_i = x_i + \frac{k}{2}$, $\overline{y}_i = y_i + \frac{k}{2}$, $\overline{z}_i = z_i + \frac{k}{2}$, $\overline{w}_i = w_i + \frac{k}{2}$ for $i = 1, 2, \ldots, \frac{k}{2}$.

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Suppose $k = 4m$, with $m \in \mathbb{N}$. Let

$$\begin{aligned} f = & \sum_{1 \le i,j \le 2m, i:odd} x_i y_j z_{[i+j-1]_{2m}} w_{[\frac{i-1}{2}+j]_{2m}} \\ & + \sum_{1 \le i,j \le 2m, i:odd} \overline{x}_i y_j \overline{z}_{[i+j-1]_{2m}} \overline{w}_{[\frac{i-1}{2}+j+m]_{2m}} \\ & + \sum_{1 \le i,j \le 2m, i:odd} x_i \overline{y}_j \overline{z}_{[i+j-1]_{2m}} \overline{w}_{[\frac{i-1}{2}+j]_{2m}} \\ & + \sum_{1 \le i,j \le 2m, i:odd} \overline{x}_i \overline{y}_j z_{[i+j-1]_{2m}} w_{[\frac{i-1}{2}+j+m]_{2m}} \end{aligned}$$

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&+ \sum_{1\le i,j\le 2m, i:even} \overline{x}_i \overline{y}_j z_{[i+j-1]_{2m}} \overline{w}_{[\frac{i-2}{2}+j+m]_{2m}}.
\end{aligned}
$$

Then f is a balanced neighborly polynomial of type (k,k,k,k).

Theorem 4

Suppose $k = 4m + 2$, with $m \in \mathbb{N}$. Let

$$\begin{aligned} f = & \sum_{1 \le i,j \le 2m+1, i:odd} x_i y_j z_{[i+j-1]_{2m+1}} w_{[\frac{i-1}{2}+j]_{2m+1}} \\ & + \sum_{1 \le i,j \le 2m+1, i:odd} \overline{x}_i y_j \overline{z}_{[i+j-1]_{2m+1}} \overline{w}_{[\frac{i-1}{2}+j+m]_{2m+1}} \\ & + \sum_{1 \le i,j \le 2m+1, i:odd} x_i \overline{y}_j \overline{z}_{[i+j-1]_{2m+1}} \overline{w}_{[\frac{i-1}{2}+j]_{2m+1}} \\ & + \sum_{1 \le i,j \le 2m+1, i:odd} \overline{x}_i \overline{y}_j z_{[i+j-1]_{2m+1}} w_{[\frac{i-1}{2}+j+m]_{2m+1}} \end{aligned}$$

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&+ \sum_{1\le j\le 2m+1} \overline{x}_{2m+1} y_j \overline{z}_j w_j + \sum_{1\le j\le 2m+1} \overline{x}_{2m+1} \overline{y}_j z_j \overline{w}_j .
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If a balanced neighborly simplicial sphere of type $(n_1, \ldots, n_d)$ exists over a field K, then balanced neighborly polynomial of type $(n_1, \ldots, n_d)$ exists over a field K.

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Corollary

There are no balanced neighborly simplicial spheres of type $(2, 2, 2, 2)$.

Technique

Lemma 3

(i) *If* f *is a squarefree polynomial of degree* d*, then* $x_1^2, \ldots, x_n^2 \in \mathrm{ann}(f)$ *and* $R/(\mathrm{ann}(f))$ *is an Artinian Gorenstein graded* K*-algebra of socle degree* d*.*

Technique

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(i) *If f is a squarefree polynomial of degree d, then $x_1^2, \dots, x_n^2 \in \operatorname{ann}(f)$ and $R/(\operatorname{ann}(f))$ is an Artinian Gorenstein graded K-algebra of socle degree d.*

(ii) *If I is a homogenerous ideal such that R/I is an Artinian Gorenstein graded K-algebra of socle degree d and $x_1^2, \dots, x_n^2 \in I$, then there is a squarefree polynomial such that $\operatorname{ann}(f) = I$.*

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Note that, an Artinian graded K-algebra $R/I = \bigoplus_{k=0}^{d} (R/I)_k$ is called *Gorenstein of socle degree d* if $0 :_{R/I} (x_1, \dots, x_n) = (R/I)_d$ has K-dimension 1, where $(R/I)_d \neq 0$.

THANK YOU FOR YOUR ATTENTION !