Let K be an infinite field. Consider the following group actions:

(A) $G := SL_d(K)$ acting on $S := K[Y_{d\times n}]$ with $M : Y \longmapsto MY$.

Then $S^G = K[\Delta : \Delta$ is a $d\times d$ minor of $Y]$ (Igusa, DeConcini–Procesi)

<u>Rmk:</u> $\operatorname{Proj} S^G = \operatorname{Grass}(d, n)$

(B) $G := GL_t(K)$ acting on $S := K[Y_{m\times t}, Z_{t\times n}]$, $M : \begin{cases} Y \longmapsto YM^{-1} \\ Z \longmapsto MZ \end{cases}$

Then $S^G = K[YZ] \simeq K[X_{m\times n}]/I_{t+1}(X)$ (D.P., Hashimoto)

(C) $G := Sp_{2t}(K) = \{M : M^{tr}\Omega M = \Omega\}$ where $\Omega = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$

acting on $S := K[Y_{2t\times n}]$ where $M : Y \longmapsto MY$.
Then $S^G = K[Y^{tr}\Omega Y] \simeq K[X^{alt}_{n\times n}]/Pf_{2t+2}(X)$ (D.P., H)

(D) $G := O_d(K) = \{M : M^{tr}M = Id\}$ acting on $S := K[Y_{d\times n}]$

with $M : Y \longmapsto MY$. Then $S^G = K[Y^{tr}Y] \simeq K[X^{sym}_{n\times n}]/I_{d+1}(X)$
(D.P.)

When $\operatorname{char} K = 0$, each G above is linearly reductive, so $R := S^G \hookrightarrow S$ is <u>pure</u>, equivalently R is a direct summand of S as an R-module.

<u>Rmk:</u> $R \to S$ is <u>pure</u> if $(\quad)\otimes_R N$ is injective $\forall$ R-mod N.

<u>Theorem</u> (HJPS). Let K be a field of char $p>0$. Then:

(A) $R := K[\Delta : \Delta\ d\times d \text{ minor}] \subseteq K[Y_{d\times n}] =: S$ is pure $\iff d = 1$ or n

<u>Rmk:</u> R is regular $\iff d = 1, n-1, n$.

(B). $R := K[YZ] \subseteq K[Y_{m\times t}, Z_{t\times n}] =: S$ is pure $\iff$ $t = 1$ or $\min\{m,n\} \leq t$.

<u>Rmk:</u> $R \simeq K[X_{m\times n}]/I_t(X)$ is regular $\iff \min\{m,n\} \leq t$

(C) $R := K[Y^{tr}\Omega Y] \subseteq K[Y_{2t\times n}] =: S$ is pure $\iff$ $n \leq t+1$

<u>Rmk:</u> $R \simeq K[X^{alt}_{n\times n}]/Pf_{2t+2}(X)$ is regular $\iff n \leq 2t+1$

(D) $R := K[Y^{tr}Y] \subseteq K[Y_{d\times n}] =: S$

is pure $\iff$

$d = 1$

$d = 2$, p odd

$p = 2$, $n \leq (d+1)/2$

p odd, $n \leq (d+2)/2$.

<u>Rmk:</u> $R \simeq K[X^{sym}_{n\times n}]/I_{d+1}$ regular $\iff n \leq d$.

<u>Proofs:</u> Via studying <u>nullcone</u> of $R \subseteq S$, i.e., $\dfrac{S}{m_R S}$

- If $R \to S$ is pure, then $(R \to S) \otimes_R H^{\dim R}_{m_R}(R)$ implies $H^{\dim R}_{m_R S}(S) \neq 0$

- Peskine-Szpiro: If S is regular, char $p>0$, and I is an ideal s.t. S/I is Cohen-Macaulay, then $H^h_I(S) = 0 \iff h = \operatorname{ht} I$.

Easiest: Case (A) $\quad \dfrac{S}{m_R S} = \dfrac{K[Y_{d\times n}]}{I_d(Y)}$ is Cohen-Macaulay (Eagon-Northcott)

If $R \to S$ is pure, then

$$\operatorname{ht} m_R S = \dim R$$

$$n-d+1 = d(n-d)+1$$

$\Rightarrow (n-d)(d-1) = 0 \quad \Rightarrow \quad d = 1$ or n.

If $d=1$, then $R = S$.
If $d = n$, then $R = K[\det Y] \subseteq S$ is free.

Case (B) $\quad \dfrac{S}{m_R S} = \dfrac{K[Y_{m\times t}, Z_{t\times n}]}{I_1(YZ)}$. This has many components:

Let $\overline{Y}, \overline{Z}$ be a point in the alg set defined by $I_1(YZ)$.

$$K^m \xleftarrow{\overline{Y}} K^t \xleftarrow{\overline{Z}} K^n$$

$\operatorname{Im} \overline{Z} \subseteq \operatorname{Ker} \overline{Y}$

$\operatorname{rank} \overline{Z} \leq t - \operatorname{rank} \overline{Y}$

$\operatorname{rank} \overline{Y} + \operatorname{rank} \overline{Z} \leq t$

The irreducible components are defined by

$$\mathfrak{p}_{ij} := I_{i+1}(Y) + I_{j+1}(Z) + I_1(YZ). \qquad i+j = t$$

- Buchsbaum-Eisenbud
- Kempf $\qquad \mathbb{C}[Y,Z]/\mathfrak{p}_{ij}$ has rat'l sing
- De Concini - Strickland
- Huneke

$\Big\}\ \dfrac{K[Y,Z]}{\mathfrak{p}_{ij}}$ is Cohen-Macaulay, K any field.

For ⓒ we prove:

<u>Theorem</u> (HJPS) $\dfrac{K[Y_{2t\times n}]}{I_1(Y^{tr}\Omega Y)}$ is a normal Cohen Macaulay domain.
K any field.

- Kraft-Schwarz (2014) $\dfrac{\mathbb{C}[Y]}{I_1(Y^{tr}\Omega Y)}$ is a normal domain

<u>Remark:</u> Let $\bar{Y}$ belong to the alg set defined by $I_1(Y^{tr}\Omega Y)$.
Then $K^n \xleftarrow{\bar{Y}^{tr}} K^{2t} \xleftarrow{\Omega} K^{2t} \xleftarrow{\bar{Y}} K^n$ is a complex.

So $\operatorname{rank} \bar{Y}^{tr} + \operatorname{rank} \bar{Y} \leq 2t$

$\Rightarrow \operatorname{rank} \bar{Y} \leq t$

<u>Nullstellensatz:</u> $I_{t+1}(Y) \subseteq$ radical $I_1(Y^{tr}\Omega Y)$

<u>Exercise:</u> $I_{t+1}(Y) \subseteq I_1(Y^{tr}\Omega Y)$.

(5) **Theorem** (HJPS) Let Y be a $d\times n$ matrix of indeterminates over a field K. Set $\mathfrak{a} := I_1(Y^{tr} Y)$ in $S := K[Y]$.

(1) Suppose $\operatorname{char} K \neq 2$ and $i = \sqrt{-1} \in K$. Then

- $\mathfrak{a}$ is radical $\iff 2n \leq d$
- $\mathfrak{a}$ is prime $\iff 2n < d$

$\Big\}$ Kraft-Schwarz (2014) if $K = \mathbb{C}$.

- If d is odd OR $2n < d$, then $\dfrac{S}{\operatorname{rad}\mathfrak{a}}$ is a Cohen-Macaulay domain

- If d is even AND $2n \geq d$, then $\mathfrak{a}$ has minimal primes P and Q, and S/P, S/Q are Cohen-Macaulay ✓

(2) Suppose $\operatorname{char} K = 2$. Then $\mathfrak{a}$ is not radical, but $S/\operatorname{rad}\mathfrak{a}$ is a Cohen-Macaulay domain. ✓

Remark: Let $\overline{Y} \in V(\mathfrak{a})$. Then $K^n \xleftarrow{\overline{Y}^{tr}} K^d \xleftarrow{\overline{Y}} K^n$ is a complex

so $2 \operatorname{rank} \overline{Y} \leq d \implies I_{\lfloor d/2 \rfloor + 1}(\overline{Y}) = 0$.

In general, $I_{\lfloor d/2 \rfloor + 1}(Y) \not\subseteq \mathfrak{a}$.

Remark: Suppose $d = 2t$. Consider $\overline{Y} := \left(\dfrac{A}{i\mathcal{O}A}\right)$ $\quad A : t\times n$, $\mathcal{O} \in O_t$

Then $\overline{Y}^{tr}\overline{Y} = \left(A^{tr} \mid i A^{tr}\mathcal{O}^{tr}\right)\left(\dfrac{A}{i\mathcal{O}A}\right) = 0$

This gives $O_t \times \mathbb{A}^{t\times n} \longrightarrow V(\mathfrak{a})$.

The components corresp to $O_t = SO_t \mathbin{\mathring{\cup}} (O_t \setminus SO_t)$