Gorensteinness for normal tangent cones of geometric ideals

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Notation

Throughout this talk, we assume

Assumption (★)

- $(A, \mathfrak{m})$: excellent normal local domain.
- $A \supset K = \overline{K}$: algebraically closed field
- $\dim A = 2$, A is *not* regular
- I is an $\mathfrak{m}$-primary integrally closed ideal
- $Q = (a, b)$ is a minimal reduction of I

$\exists$ resolution of sing. $f\colon X \to \operatorname{Spec} A$, $\exists\, Z$ an anti-nef cycle on X s.t.

$$I\mathcal{O}_X = \mathcal{O}_X(-Z), \qquad I = H^0(X, \mathcal{O}_X(-Z)).$$

Then we denote it by $I = I_Z$.

(So, $I = I_Z$ means that I is an $\mathfrak{m}$-primary integrally closed ideal)

Definition 1.1 (Normal reduction numbers (cf. [OWY4]))

- The relative normal reduction number of I is defined by

$$\mathbf{nr}(I) := \min\{r \geq 1 \mid \overline{I^{r+1}} = Q\overline{I^r}\}.$$

- The normal reduction number of I is defined by

$$\bar{\mathrm{r}}(I) := \min\{r \geq 1 \mid \overline{I^{N+1}} = Q\overline{I^N}\ (\forall N \geq r)\}.$$

Then $1 \leq \mathbf{nr}(I) \leq \bar{\mathrm{r}}(I)$.

For any given $r \geq 3$, $\exists I = I_Z \subset A$: s.t. $1 = \mathbf{nr}(I) < r = \bar{\mathrm{r}}(I)$ (cf. [OWY5]).

q(nI)

For any $I = I_Z$ and a positive integer n, we put

$$q(nI) := \dim_K H^1(\mathcal{O}_X(-nZ)).$$

$p_g(A) := \dim_K H^1(\mathcal{O}_X) = q(0I)$ is called a geometric genus.

Proposition 1.2 (cf. [OWY1,OWY4])

For any minimal reduction Q of I,

(1) $0 \leq q(I) \leq p_g(A)$.

(2) $k \geq 0 \implies q(kI) \geq q((k+1)I)$.

(3) $q(nI) = q((n+1)I) \implies q(nI) = q(mI)\ (\forall m \geq n)$.

Put $q(\infty I) = q(nI)$ for large enough n.

(4) $2 \cdot q(kI) + \ell_A(\overline{I^{k+1}}/Q\overline{I^k}) = q((k+1)I) + q((k-1)I)$.

$\bar{\mathrm{r}}(I)$ and $q(nI)$

Proposition 1.3 ([OWY5, Proposition 2.2])

For $I = I_Z$, we have

(1) $\mathrm{nr}(I) = \min\{n \in \mathbb{Z}_+ \mid q((n-1)I) - q(nI) = q(nI) - q((n+1)I)\}$.

(2) $\bar{\mathrm{r}}(I) = \min\{n \in \mathbb{Z}_+ \mid q((n-1)I) = q(nI)\}$.

(3) $\bar{\mathrm{r}}(I) \leq p_g(A) + 1$. *If equality holds, then* $\mathrm{nr}(I) = 1$.

For instance, if $\bar{\mathrm{r}}(I) = 2$, then $q(2I) = q(I)$ holds. Hence

$$2 \cdot q(I) + \ell_A(\overline{I^2}/QI) = q((2I) + q(0I)$$

$$\Longrightarrow \quad \ell_A(\overline{I^2}/QI) = p_g(A) - q(I)$$

■ $\exists I = I_Z \subset A$ s.t. $p_g(A) = 2 > q(I) = 1 > q(2I) = \cdots q(\infty I) = 0$.

So $1 = \mathrm{nr}(I) < \bar{\mathrm{r}}(I) = 3$ and $p_g(A) = 2$.

Normal Hilbert coefficients

Proposition 1.4 ([OWY2, Theorem 3.2])

The normal Hilbert polynomial $\bar{P}_I(n)$ can be written as the folloiwing form:

$$\bar{P}_I(n) = \bar{e}_0(I)\binom{n+2}{2} - \bar{e}_1(I)\binom{n+1}{1} + \bar{e}_2(I)$$

(1) $\overline{P_I}(n) = \ell_A(A/\overline{I^{n+1}})$ $(\forall n \geq p_g(A) - 1)$.

(2) $\bar{e}_0(I) = e_0(I) = -Z^2$.

(3) $\bar{e}_1(I) - \bar{e}_0(I) + \ell_A(A/I) = p_g(A) - q(I)$.

(4) $\bar{e}_2(I) = p_g(A) - q(nI) = p_g(A) - q(\infty I)$ $(\forall n \geq p_g(A))$.

- *Each $\bar{e}_i(I)$ is called a normal Hilbert coefficient.*

Some graded algebra (normal tangent cone etc.)

Definition 1.5

For $I = I_Z$,

1. $G(I) := \bigoplus_{n \geq 0} I^n / I^{n+1}$: the associted graded ring of I.
2. $\overline{G}(I) := \bigoplus_{n \geq 0} \overline{I^n} / \overline{I^{n+1}}$: the normal tangent cone of I.

Question: When is $\overline{G}(I)$ Gorenstein

In this talk, we consider the following question for ‘geometric’ ideals.

Question 1.6

1. When is $\overline{G}(I)$ Cohen-Macaulay?
2. When is $\overline{G}(I)$ Gorenstein?

The above question is related to our previous reserach, which the following observation shows.

Proposition 1.7

If $\overline{G}(I)$ is Cohen-Macaulay, then $\mathbf{nr}(I) = \bar{\mathbf{r}}(I)$.

Example: $\overline{G}(I)$ is not CM

Ex 1.8

Let K be a filed of $\mathrm{char}K \neq 2, 3$, and let $A = K[[xy, xz, y^2, yz, z^2]]$ with $x^2 = y^6 + z^6$. That is, $A = K[a, b, c, d, e]/\mathfrak{a}$, where $\mathfrak{a}$ is generated by the following polynomials:

$$\begin{array}{lll} a^2 - d^4 - de^3, & ab - cd^3 - ce^3, & b^2 - d^3e - e^4 \\ ac - bd, & ae - bc, & de - c^2 \end{array}$$

Then A is an excellent normal local domain with $v = e + d - 1 = 5$.

If we put $I = (xy, xz, yz, y^2, z^4) = (a, b, c, d, e^2) \supset Q = (c, d - e^2)$, then $I = I_Z$ and $1 = \mathrm{nr}(I) < \bar{\mathrm{r}}(I) = 3$.

In particular, $\overline{G}(I)$ is NOT Cohen-Macaulay.

On the other hand, $G(I)$ is Cohen-Macaulay.

Definition 1.9 ([OWY1])

An ideal $I = I_Z$ is called a p_g-ideal if it satisfies one of the following equivalent conditions:

1. $\bar{r}(I) = 1$.
2. $q(I) = p_g(A)$.
3. $\bar{e}_1(I) - \bar{e}_0(I) + \ell_A(A/I) = 0$.
4. $\bar{e}_2(I) = 0$.
5. I is *normal* and stable.

- $I = I_Z$ is normal if $\overline{I^n} = I^n$ for every $n \geq 1$.
- $I = I_Z$ is stable if $I^2 = QI$ for some minimal reduction Q of I.

Proposition 1.10

A is a rational singularity (i.e. $p_g(A) = 0$) $\Longleftrightarrow$ $\forall I = I_Z$ is a p_g-ideal.

$\overline{G}(I)$ of p_g-ideals

Proposition 1.11 ([ORWY])

If I is a p_g-ideal, then

1. $\overline{G}(I) = G(I)$ is *Cohen-Macaulay*.
2. $\overline{\mathcal{R}}(I) = \mathcal{R}(I)$ is a Cohen-Macaulay normal domain.

An ideal $I = I_Z$ is called good if I is stable with $I = Q : I$.

Theorem 1.12

Suppose that I is a p_g-ideal. Then the following conditions are equivalent:

1. $\overline{G}(I) = G(I)$ is *Gorenstein*.
2. I is *good*.

Elliptic ideal

Definition 1.13 ([OWRY, Theorem 3.2])

An ideal $I = I_Z$ is called a elliptic ideal if it satisfies one of the following equivalent conditions:

1. $\bar{\mathrm{r}}(I) = 2$.
2. $p_g(A) > q(I) = q(\infty I)$.
3. $\bar{e}_1(I) - \bar{e}_0(I) + \ell_A(A/I) = \bar{e}_2(I) > 0$.

Theorem 1.14 ([ORWY, Corollary 3.7])

A is an elliptic singularity $\Longrightarrow \bar{\mathrm{r}}(I) \leq 2$ *for any* $I = I_Z$.

How about the converse?

Main Question

Question 1.15

Assume that I is an elliptic ideal.

1. When is $\overline{G}(I)$ Cohen-Macaulay?
2. When is $\overline{G}(I)$ Gorenstein?

Theorem 1.16 ([ORWY], Huneke)

If I is elliptic ideal, then $\overline{G}(I)$ is Cohen-Macaulay.

Main result

The following theorem gives a characterization for Gorensteinness of $\overline{G}(I)$ for any elliptic ideal I.

Theorem 2.1

Assume A is Gorenstein and $\bar{r}(I) = 2$.

Then the following conditions are equivalent:

1. *$\overline{G}(I)$ is Gorenstein.*
2. *$Q : I = Q + \overline{I^2}$ holds true.*
3. *$\ell_A(\overline{I^2}/QI) = \ell_A(A/I)$ holds true.*
4. *$\overline{e}_2(I) = \ell_A(A/I)$ holds true.*
5. *$KZ = -Z^2$, that is, $\chi(Z) = 0$.*

Sketch of the proof $(2) \Longrightarrow (1)$

Theorem **1.16** $\Longrightarrow \overline{G} := \overline{G}(I)$ is Cohen-Macaulay.

Then $a^*, b^* \in \overline{G}$ forms a $\overline{G}$-sequence, where $Q = (a, b)$.

If we put $B = \overline{G}/(a^*, b^*) \cong A/I \oplus I/(Q + \overline{I^2}) \oplus (Q + \overline{I^2})/Q$, then

$$\overline{G}\text{: Gorenstein} \iff B\text{: Gorenstein.}$$

$(2) \Longrightarrow (1)$: **ETS**: $\dim_K \mathrm{Soc}(B) = 1$.

Let $x^* \in \mathrm{Soc}(B)$, a homogeneous element.

When $x^* \in B_0$, $x^* B_2 = 0 \Rightarrow x \in Q : (Q + \overline{I^2}) \overset{(2)}{=} Q : (Q : I) = I$.

When $x^* \in B_1$, $x^* B_1 = 0 \Rightarrow x \in Q : I = Q + \overline{I^2}$.

Hence $\mathrm{Soc}(B) \subset \mathrm{Soc}(B_2) \cong K$, as required.

(1) $\overline{G}(I)$ is Gorenstein.

(2) $Q : I = Q + \overline{I^2}$.

Sketch of the proof $(1) \Longrightarrow (2) \Longleftrightarrow (3)$

(Obs i) Huneke-Itoh's theorem (i.e. $Q \cap \overline{I^2} = QI$)

$\Longrightarrow \ell_A(B_2) = \ell_A(Q + \overline{I^2}/Q) = \ell_A(\overline{I^2}/Q \cap \overline{I^2}) = \ell_A(\overline{I^2}/QI)$.

(Obs ii) Matlis duality

$\Longrightarrow \ell_A(Q : I/Q) = \ell_A(K_{A/I}) = \ell_A(A/I) = \ell_A(B_0)$.

(Obs iii) $\bar{\mathrm{r}}(I) = 2 \Longrightarrow I\overline{I^2} \subset \overline{I^3} = Q\overline{I^2} \Longrightarrow Q + \overline{I^2} \subset Q : I$.

$$\therefore \ell_A(B_0) - \ell_A(B_2) = \ell_A(Q : I/Q + \overline{I^2}) = \ell_A(\overline{I^2}/QI) - \ell_A(A/I).$$

Hence this implies $(1) \Longrightarrow (2) \Longleftrightarrow (3)$.

(1) $\overline{G}(I)$ is Gorenstein.

(2) $Q : I = Q + \overline{I^2}$.

(3) $\ell_A(\overline{I^2}/QI) = \ell_A(A/I)$.

Assume $\bar{\mathrm{r}}(I) = 2$. Then $q(I) = q(2I) = q(\infty I)$.

(Obs i) Prop. 1.2 $\Longrightarrow \ell_A(\overline{I^2}/QI) = p_g(A) - q(I)$.

(Obs ii) Prop. 1.4 $\Longrightarrow \bar{e}_2(I) = p_g(A) - q(\infty I) \left(= \ell_A(\overline{I^2}/QI)\right)$.

(Obs iii) Kato's Riemann-Roch formula

$$\underline{\ell_A(A/I) + q(I) = \chi(Z) + p_g(A)}, \text{ where } \chi(Z) = -\frac{Z^2 + KZ}{2}.$$

$$\Longrightarrow \chi(Z) = \ell_A(A/I) - \left\{p_g(A) - q(I)\right\}.$$

Hence $\quad \ell_A(A/I) - \ell_A(\overline{I^2}/QI) = \ell_A(A/I) - \bar{e}_2(I) = \chi(Z)$.

(3) $\ell_A(\overline{I^2}/QI) = \ell_A(A/I)$.

(4) $\bar{e}_2(I) = \ell_A(A/I)$.

(5) $\chi(Z) = 0$, that is, $KZ = -Z^2$.

Strongly elliptic ideal

Definition 2.2 ([OWRY, Theorem 3.9])

An ideal $I = I_Z$ is called a strongly elliptic ideal if it satisfies one of the following equivalent conditions:

1. $\bar{\mathrm{r}}(I) = 2$ and $\ell_A(\overline{I^2}/QI) = 1$ for some min. reduction Q of I.
2. $p_g(A) - 1 = q(I) = q(\infty I)$.
3. $\bar{e}_2(I) = 1$.

A (resp. A Gorenstein) local ring A is called a strongly elliptic singularity (resp. minimally elliptic singularity) if $p_g(A) = 1$.

Theorem 2.3 ([ORWY, Theorem 3.14])

A is a strong elliptic singularity

$\Rightarrow$ any $I = I_Z$ is either a p_g-ideal or a strongly elliptic ideal.

When is $\overline{G}(\mathfrak{m})$ Gorenstein

In what follows, we always assume that A is Gorenstein.

When is $\overline{G}(\mathfrak{m})$ Gorenstein?

Proposition 2.4

1. $\bar{r}(\mathfrak{m}) \le 2 \Longrightarrow \overline{G}(\mathfrak{m})$ *is Gorenstein.*
2. *If* $p_g(A) \le 2 \Longrightarrow \overline{G}(\mathfrak{m})$ *is Gorenstein.*

If $\bar{r}(\mathfrak{m}) = 1$, then $\mathfrak{m}$ is good.

If $\bar{r}(\mathfrak{m}) = 2$, then $\mathfrak{m}$ satisfies the condition of the theorem.

$p_g(A) = 2$ (A:Gor.) $\overset{\textit{Yau}}{\Longrightarrow}$ A: elliptic $\overset{\textit{Okuma}}{\Longrightarrow}$ $\bar{r}(I) \le 2$ for $\forall I = I_Z$.

Lemma 2.5

$A = K[[x, y, z]]/(x^a + y^b + z^c)$

$$\Longrightarrow p_g(A) = \sum_{n=0}^{a(A)} \dim_K A_n$$

For example, we consider $A = \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^{6g+1})$.

Put $\deg(x) = 3(6g+1)$, $\deg(y) = 2(6g+1)$, and $\deg(z) = 6$.

Then $a(A) = 6(6g+1) - \{3(6g+1) + 2(6g+1) + 6\} = 6g - 5$.

$$\therefore p_g(A) = \sum_{n=0}^{6g-5} \dim_K A_n = \sharp\left\{1, z, z^2, \ldots, z^{g-1}\right\} = g.$$

When is $\overline{G}(\mathfrak{m})$ Gorenstein: Examples (1)

Ex 2.6

Let $g \geq 1$ be an integer.

1. $A = \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^{6g+1})$.

   $p_g(A) = g$, $\bar{r}(\mathfrak{m}) = 1$,

   $\overline{G}(\mathfrak{m}) = G(\mathfrak{m}) \cong K[X, Y]/(X^2)$ is Gorenstein.

2. $A = \mathbb{C}[[x, y, z]]/(x^2 + y^4 + z^{4g})$

   $p_g(A) = g$, $\bar{r}(\mathfrak{m}) = 2$, $\overline{G}(\mathfrak{m})$ is Gorenstein. Indeed,

$$\overline{G}(\mathfrak{m}) \cong \begin{cases} \mathbb{C}[X, Y, Z]/(X^2 + Y^4 + Z^4) & (\text{if } g = 1) \\ \mathbb{C}[X, Y, Z]/(X^2 + Y^4) & (\text{o.e.}) \end{cases}$$

When is $\overline{G}(\mathfrak{m})$ Gorenstein: Examples (2)

1. $\exists A$ which satisfies $\bar{\mathrm{r}}(\mathfrak{m}) = 3$ and $\overline{G}(\mathfrak{m})$ is Not Gorenstein.
2. $\exists A$ which satisfies $p_g(A) = 3$ and $\overline{G}(\mathfrak{m})$ is Not Gorenstein.

Ex 2.7

$A = \mathbb{C}[[x, y, z]]/(x^3 + y^5 + z^5)$.

Then $\bar{\mathrm{r}}(\mathfrak{m}) = p_g(A) = 3$, and that $\overline{G}(\mathfrak{m})$ is Not Gorenstein

Application: Elliptic ideals in Gorenstein elliptic singularity

Using geometric tools, we give the following finiteness therorem.

Theorem 2.8

Assume that A is an elliptic Gorenstein singularity.

If we put $\mathcal{G}_{ell} := \{I = I_Z \subset A \mid I \text{ is elliptic}, \overline{G}(I) \text{ is Gorenstein}\}$,

then we have $\sharp(\mathcal{G}_{ell}) \leq p_g(A)$.

A is called an elliptic singularuty if $\chi(D) \geq 0$ for any cycle $D > 0$ and $\chi(F) = 0$ for some cycle $F > 0$.

Question 2.9

For any Gorentein local ring A, is the set $\mathcal{G}_{ell}$ above a finite set?

Ex 2.10

Let $A = \mathbb{C}[[x, y, z]]/(x^2 + z(z^{4n+2} + y^4))$ and

$$B = \mathbb{C}[x, y, z]/(x^2 + y^3 + z^{6(2n+1)}).$$

Then both A and B are elliptic Gorenstein singularities with the same resolution graph.

1. $p_g(A) = n + 1$ and $\sharp(\mathcal{G}_{ell}) = n$.
   In fact, $\mathcal{G}_{ell} = \{(x, y, z^j) \mid j = 1, 2, \dots, n\}$.
2. $p_g(B) = 2n + 1$ and $\mathcal{G}_{ell} = \emptyset$.

Note: $\overline{G}(\mathfrak{m})$ is Gorenstein. But $\mathfrak{m} \subset B$ is not an elliptic ideal but a good p_g-ideal.

Thank you very much for your attention !

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