On the Hilbert coefficients of graded modules over graded rings

Koji Nishida (Chiba University)

The 43rd Symposium on Commutative Algebra

November 14, 2022

1. Introduction

Throughout this talk,

- $R = \oplus_{n\in\mathbb{N}} R_n$ is a Noetherian $\mathbb{N}$-graded ring such that R_0 is an Artinian local ring and $R = R_0[R_1]$, where $\mathbb{N} = \{0, 1, 2, \dots\}$.
- $M = \oplus_{n\in\mathbb{N}} M_n$ is a finitely generated $\mathbb{N}$-graded R-module.

We set $d := \dim R$ and $s := \dim_R M$.
As is well known, $\exists\, \mathrm{e}_i(M) \in \mathbb{Z}$ for $i = 0, 1, \dots, s$ such that

$$\sum_{k=0}^{n} \ell_{R_0}(M_k) = \sum_{i=0}^{s} (-1)^i \cdot \mathrm{e}_i(M) \cdot \binom{n+s-i}{s-i}$$

for $n \gg 0$. We call $\mathrm{e}_i(M)$ the i-th Hilbert coefficient of M.

In particular, $\mathrm{e}_0(M)$ is the multiplicity of M, which is a classical and useful invariant of M. Although many important facts on $\mathrm{e}_0(M)$ are known, let us focus our attention on the following well-known result.

Let $f_1, f_2, \ldots, f_s \in R_1$ be an sop for M. Then we have

$$\mathrm{e}_0(M) \leqq \ell_R(\, M/(f_1, f_2, \ldots, f_s)M\,)$$

and the equality holds if and only if M is a CM R-module.

The purpose of my talk is to give a generalization of this result, which is an assertion on $\mathrm{e}_i(M)$ for $\forall i \geqq 0$.
We will see that Hilbert coefficients have depth sensitivity.

2. Definition of Hilbert coefficients

Let t be an indeterminate. We set $\Delta := 1 - t$ and

$$\mathrm{P}_M := \sum_{n \in \mathbb{N}} \ell_{R_0}(M_n) \cdot t^n \in \mathbb{Z}[[t]].$$

Then we have $\Delta^s \cdot \mathrm{P}_M \in \mathbb{Z}[t]$.

Definition 2.1 (Hilbert coefficients of M)

$$\text{For } \forall i \in \mathbb{Z},\ \mathrm{e}_i(M) := \begin{cases} \dfrac{1}{i!} \cdot \dfrac{d^i}{dt^i}(\Delta^s \cdot \mathrm{P}_M)\Big|_{t=1} & \text{if } i \geqq 0, \\ 0 & \text{if } i < 0. \end{cases}$$

It is easy to see that $\max\{i \in \mathbb{Z} \mid \mathrm{e}_i(M) \neq 0\} = \deg \Delta^s \cdot \mathrm{P}_M$ and

$$(\sharp) \quad \Delta^s \cdot \mathrm{P}_M = \sum_{i \in \mathbb{N}} \mathrm{e}_i(M) \cdot (t-1)^i ,$$

which is the Taylor series expansion around $t = 1$. Let us notice that

$$t - 1 = -\Delta \text{ and } \Delta^{-1} = 1 + t + t^2 + \cdots \in \mathbb{Z}[[\, t \,]] .$$

Hence, multiplying $\Delta^{-(s+1)}$ to the both sides of $(\sharp)$, we get

$$\Delta^{-1} \cdot \mathrm{P}_M = \sum_{i \in \mathbb{N}} (-1)^i \cdot \mathrm{e}_i(M) \cdot \Delta^{-(s-i+1)} .$$

Then, comparing the coefficients of the term of t^n, we see that

$$\sum_{k=0}^{n} \ell_{R_0}(\, M_k \,) = \sum_{i=0}^{s} (-1)^i \cdot \mathrm{e}_i(M) \cdot \binom{n+s-i}{s-i}$$

holds for $n \gg 0$.

3. How to compute Hilbert coefficients

Proposition 3.1

For $\forall i\,, \forall r \in \mathbb{N}$, we have

$$\mathrm{e}_i(M(-r)) \;=\; \sum_{j=0}^{\min\{i,r\}} \binom{r}{j} \cdot \mathrm{e}_{i-j}(M)\,,$$

where $M(-r)$ denotes the $\mathbb{N}$-graded R-module with grading given by

$$[\,M(-r)\,]_n = \begin{cases} M_{n-r} & \text{if } n \geqq r \\ 0 & \text{if } n < r\,. \end{cases}$$

In particular, for $0 < \forall r \in \mathbb{N}$, we have

$$\mathrm{e}_0(M(-r)) = \mathrm{e}_0(M) \quad \text{and} \quad \mathrm{e}_1(M(-r)) = \mathrm{e}_1(M) + r \cdot \mathrm{e}_0(M)\,.$$

Proposition 3.2

Let $0 < r \in \mathbb{N}$ and $f \in R_r$ be M-regular. Then, for $\forall i \in \mathbb{N}$,

$$\mathrm{e}_i(M/fM) = \sum_{j=1}^{\min\{i+1, r\}} \binom{r}{j} \cdot \mathrm{e}_{i-j+1}(M).$$

In particular, if $f \in R_1$ is M-regular, $\mathrm{e}_i(M/fM) = \mathrm{e}_i(M)$ for $\forall i \in \mathbb{N}$.

Proposition 3.3

Let R be a polynomial ring over a field with d variables of degree 1. If M has an $\mathbb{N}$-graded R-free resolution

$$0 \longrightarrow F^{(\ell)} \longrightarrow \cdots \longrightarrow F^{(1)} \longrightarrow F^{(0)} \longrightarrow 0$$

such that $F^{(j)} \cong \oplus_{r \in \mathbb{N}} R(-r)^{\beta_{jr}}$ for $\forall j = 0, 1, \ldots, \ell$, then

$$\mathrm{e}_i(M) = \sum_{r=i+d-s}^{\infty} \left\{ \sum_{j=0}^{\ell} (-1)^{d-s+j} \cdot \beta_{jr} \right\} \cdot \binom{r}{i+d-s}$$

holds for $\forall i \in \mathbb{N}$.

Proposition 3.4

Suppose that a filtration

$$0 = M^{(0)} \subsetneqq M^{(1)} \subsetneqq M^{(2)} \subsetneqq \cdots \subsetneqq M^{(\ell)} = M$$

of $\mathbb{N}$-graded R-modules of M is given. For $\forall j = 1, 2, \ldots, \ell$, we set

$$s_j := \dim_R M^{(j)}/M^{(j-1)} .$$

Let us take $\forall i \in \mathbb{N}$ and set

$$\Lambda_i := \{ j \in \mathbb{N} \mid 1 \leqq j \leqq \ell \text{ and } s_j \geqq s - i \} .$$

Then we have

$$\mathrm{e}_i(M) = \sum_{j \in \Lambda_i} (-1)^{s-s_j} \cdot \mathrm{e}_{s_j - s + i}(M^{(j)}/M^{(j-1)}) .$$

4. M-filter-regular sequence

Definition 4.1 (N. V. Trung)

An element $f \in R_1$ is said to be M-filter-regular if $\ell_R(\, 0 :_M f\,) < \infty$.
A sequence of elements $f_1, \dots, f_r \in R_1$ is said to be M-filter-regular if

$$f_i \text{ is } M/(f_1, \dots, f_{i-1})M\text{-filter-regular for } \forall i = 1, \dots, r\,,$$

where $(f_1, \dots, f_{i-1})M$ denotes the zero module when $i = 1$.

Of course, any M-regular sequence is an M-filter-regular sequence.
On the other hand, if $f_1, \dots, f_s \in R_1$ form an sop for M, then $(f_1, \dots, f_s)R$ can be generated by an M-filter-regular sequence.

5. Depth sensitivity of Hilbert coefficients

Theorem 5.1

Let $s > 0$ and $0 \leq i < s$. If we choose $f_1, \ldots, f_{s-i} \in R_1$ so that they form an M-filter-regular sequence, we have

$$\dim_R M/(f_1, \ldots, f_{s-i})M = i$$

and the following assertions hold.

(1) If i is even, then $\mathrm{e}_i(M) \leq \mathrm{e}_i(M/(f_1, \ldots, f_{s-i})M)$.

(2) If i is odd, then $\mathrm{e}_i(M) \geq \mathrm{e}_i(M/(f_1, \ldots, f_{s-i})M)$.

(3) $\mathrm{e}_i(M) = \mathrm{e}_i(M/(f_1, \ldots, f_{s-i})M) \Leftrightarrow \operatorname{depth}_R M \geq s - i$.

6. Examples

In this section, let $R = K[x, y, z, w]$ be a polynomial ring over a field, which is a graded ring such that $\deg x = \deg y = \deg z = \deg w = 1$.

Set $\mathfrak{m} := (x, y, z, w)R$ and $A := R/\mathfrak{p}$, where $\mathfrak{p} = \mathrm{I}_2 \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$.

Example 6.1

(1) A is Cohen-Macaulay, $\dim_R A = 2$, and x, w is A-regular.

(2) $\mathrm{e}_0(A) = 3$, $\mathrm{e}_1(A) = 2$, and $\mathrm{e}_i(A) = 0$ for $\forall i \geqq 2$.

In fact, by the theorem of Hilbert-Burch, A has an R-free resolution

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow A \longrightarrow 0.$$

Moreover, $(x, w)R + \mathfrak{p} = (x, y^2, z^2, w)R$ is $\mathfrak{m}$-primary.

Next, we set $B := R/\mathfrak{mp}$.

Example 6.2

(1) $\dim_R B = 2$ and $\operatorname{depth}_R B = 0$.

(2) x, w is a B-filter-regular sequence.

(3) $\mathrm{e}_0(B) = 3$, $\mathrm{e}_1(B) = 2$ and $\mathrm{e}_2(B) = 3$.

(4) $\mathrm{e}_0(B/(x,w)B) = 6$ and $\mathrm{e}_1(B/xB) = -1$.

In fact, we get (1), (2) and (3) by considering the exact sequence

$$0 \longrightarrow \mathfrak{p}/\mathfrak{mp} \longrightarrow B \longrightarrow A \longrightarrow 0 .$$

Moreover, we get (4) applying Proposition 3.4 to the filtration

$$0 \subsetneq (x,w)B/xB \subsetneq B/xB ,$$

whose quotient modules can be studied easily as we have

$$(x,w)B/xB \cong (A/xA)(-1) \quad \text{and} \quad B/(x,w)B \cong K[y,z]/(y,z)^3 .$$

Finally, we set $C := R/\mathfrak{p}\mathfrak{q}$, where $\mathfrak{q} = (x, y)R$, and $f := x + y + z$.

Example 6.3

(1) $\dim_R C = 2$ and $\operatorname{depth}_R C = 1$.

(2) w, f is a C-filter-regular sequence.

(3) $\mathrm{e}_0(C) = 4$, $\mathrm{e}_1(C) = 2$ and $\mathrm{e}_2(C) = -3$

(4) $\mathrm{e}_0(C/(w, f)C) = 6$ and $\mathrm{e}_1(C/wC) = 2$.

In fact, we get (3) applying Proposition 3.4 to the filtration

$$0 \subsetneq xC \subsetneq \mathfrak{q}C \subsetneq C,$$

whose quotient modules can be studied easily as we have

$$xC \cong A(-1), \ \mathfrak{q}C/xC \cong (A/xA)(-1), \ C/\mathfrak{q}C \cong K[y, z]/(y, z)^3.$$

We get $\mathrm{e}_0(C/(w, f)C) = 6$ as $C/(w, f)C \cong K[y, z]/(y, z)^3$.
For computing $\mathrm{e}_1(C/wC)$, we use $0 \subsetneq (x, w)C/wC \subsetneq C/wC$.

Thank you.