On the Ehrhart ring of the stable set polytope of a cycle graph

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For sets X, Y,

$\#X$: the cardinality of X.

$Y^X := \{f \mid f\colon X \to Y\}$.

For a finite set X, we identify $\mathbb{R}^X$ with $\mathbb{R}^{\#X}$, the Euclidean space.

For f, f_1, $f_2 \in \mathbb{R}^X$ and $a \in \mathbb{R}$, we define maps $f_1 \pm f_2$ and af by

$$(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$$

$$(af)(x) = a(f(x))$$

for $x \in X$.

For a subset A of X, we define the characteristic function $\chi_A \in \mathbb{R}^X$ by

$$\chi_A(x) = 1 \text{ for } x \in A \text{ and } \chi_A(x) = 0 \text{ for } x \in X \setminus A.$$

For a nonempty subset $\mathscr{X}$ of $\mathbb{R}^X$, we define

$$\operatorname{conv}\mathscr{X} := (\text{the convex hull of } \mathscr{X}),$$

$$\operatorname{aff}\mathscr{X} := (\text{affine span of } \mathscr{X})$$

$$\operatorname{relint}\mathscr{X} := (\text{the interior of } \mathscr{X} \text{ in the topological space } \operatorname{aff}\mathscr{X}).$$

Definition 1 Let X be a finite set and $\xi \in \mathbb{R}^X$. For $B \subset X$, we set $\xi^+(B) := \sum_{b \in B} \xi(b)$. We define the empty sum to be 0, i.e., $\xi^+(\emptyset) = 0$.

In this talk, all graphs are finite simple graphs without loop.

For a graph G with vertex set V and edge set E we denote $G = (V, E)$ or $V = V(G)$ and $E = E(G)$.

If $\{a, b\} \in E$, where a, $b \in V$, we say that a and b are adjacent.

A clique of G is a subset K of V such that any two elements of K are adjacent.

If v_1, v_2, $\ldots$, v_r are distinct vertices of G with $r \geq 3$, $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq r-1$ and $\{v_r, v_1\} \in E$, then we say that $v_1 v_2 \cdots v_r v_1$ is a cycle (of length r). A cycle with even (resp. odd) length is called an even (resp. odd) cycle.

Suppose that $v_1 v_2 \cdots v_r v_1$ is a cycle. If $\{v_i, v_j\} \in E$ and $2 \leq |i - j| \leq r - 2$, we say that $\{v_i, v_j\}$ is a chord of the cycle $v_1 v_2 \cdots v_r v_1$.

Definition 2 If a graph G consists of one cycle without chord, we say that G is a cycle graph.

Definition 3 $S \subset V$ is called a stable set if $\{a, b\} \notin E$ for any a, $b \in S$. We set

$$\mathrm{STAB}(G) := \mathrm{conv}\{\chi_S \in \mathbb{R}^V \mid S \text{ is a stable set of } G\}$$

and call the stable set polytope of G.

Remark 4 It is clear that for $f \in \mathrm{STAB}(G)$,

(1) $0 \leq f(x) \leq 1$ for any $x \in V$.

(2) $f^+(K) \leq 1$ for any clique K in G.

(3) $f^+(C) \leq \frac{\#C-1}{2}$ for any odd cycle C.

Definition 5 We set

$$\mathrm{TSTAB}(G) := \left\{ f \in \mathbb{R}^V \;\middle|\; \begin{array}{l} f \text{ satisfies (1) and (3) above and } f^+(e) \leq \\ 1 \text{ for any } e \in E \end{array} \right\}.$$

If $\mathrm{STAB}(G) = \mathrm{TSTAB}(G)$, then G is called a t-perfect graph.

Remark 6 $\mathrm{STAB}(G) \subset \mathrm{TSTAB}(G)$.

Fact 7 Every cycle graph is t-perfect.

$\mathbb{K}$: a field.

X: a finite set.

$\mathscr{P}$: a rational convex polytope in $\mathbb{R}^X$.

$-\infty$: a new element with $-\infty \notin X$.

$X^- := X \cup \{-\infty\}$.

$\{T_x\}_{x \in X^-}$: a family of indeterminates indexed by X^-.

For $f \in \mathbb{Z}^{X^-}$, we denote the Laurent monomial $\prod_{x \in X^-} T_x^{f(x)}$ in $\mathbb{K}[T_x^{\pm 1} \mid x \in X^-]$ by T^f.

Set $\deg T_x = 0$ for $x \in X$ and $\deg T_{-\infty} = 1$.

Definition 8 The Ehrhart ring of $\mathscr{P}$ over a field $\mathbb{K}$ is the subring

$$\mathbb{K}[T^f \mid f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f|_X \in \mathscr{P}]$$

of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in X^-]$.

We denote the Ehrhart ring of $\mathscr{P}$ over $\mathbb{K}$ by $E_{\mathbb{K}}[\mathscr{P}]$.

Fact 9 $E_{\mathbb{K}}[\mathscr{P}]$ is a Noetherian normal and Cohen-Macaulay domain.

Remark 10 $\dim E_{\mathbb{K}}[\mathscr{P}] = \dim \mathscr{P} + 1$.

Fact 11 The ideal

$$\bigoplus_{f\in\mathbb{Z}^{X^-}, f(-\infty)>0, \frac{1}{f(-\infty)} f|_X \in \operatorname{relint}\mathscr{P}} \mathbb{K}T^f$$

of $E_{\mathbb{K}}[\mathscr{P}]$ is the canonical module of $E_{\mathbb{K}}[\mathscr{P}]$.

We denote the ideal of Fact 11 by $\omega_{E_{\mathbb{K}}[\mathscr{P}]}$ and call the canonical ideal of $E_{\mathbb{K}}[\mathscr{P}]$.

Definition 12 Let R be a commutative ring and M an R-module. We set

$$\mathrm{tr}(M) := \sum_{\varphi \in \mathrm{Hom}(M,R)} \varphi(M)$$

and call $\mathrm{tr}(M)$ the trace of M.

Fact 13 (Herzog-Hibi-Stamate) Let R be a Cohen-Macaulay local or graded ring over a field with canonical module ω_R. Then for $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ is Gorenstein if and only if $\mathfrak{p} \not\supset \mathrm{tr}(\omega_R)$. In particular, R is Gorenstein if and only if $\mathrm{tr}(\omega_R) = R$.

Fact 14 (Ohsugi-Hibi, Hibi-Tsuchiya) Let $G = (V, E)$ be a cycle graph. Then the Ehrhart ring $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ of the stable set polytope of G is Gorenstein if and only if the length of the cycle V is even or less than 7.

In the rest of this talk, we assume that $G = (V, E)$ is an odd cycle graph with length at least 7.

We set $V = \{v_0, v_1, \ldots, v_{2\ell}\}$, where ℓ is an integer with $\ell \geq 3$ and $E = \{\{v_i, v_{i+1}\} \mid 0 \leq i \leq 2\ell - 1\} \cup \{\{v_{2\ell}, v_0\}\}$.

Further, We set $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i \leq 2\ell - 1$ and $e_{2\ell} = \{v_{2\ell}, v_0\}$ and $R = E_{\mathbb{K}}[\mathrm{STAB}(G)]$. Then

$$\begin{aligned} \mathrm{STAB}(G) &= \mathrm{TSTAB}(G) \\ &= \left\{ \nu \in \mathbb{R}^V \;\middle|\; \begin{array}{l} \nu(v_i) \geq 0,\ \nu^+(e_i) \leq 1 \text{ for } 0 \leq i \leq 2\ell \text{ and} \\ \nu^+(V) \leq \ell \end{array} \right\}. \end{aligned}$$

Definition 15 For $n \in \mathbb{Z}$, we set

$$t\mathcal{U}^{(n)} := \left\{ \mu \in \mathbb{Z}^{V^-} \;\middle|\; \begin{array}{l} \mu(x) \geq n \text{ for any } x \in V, \\ \mu^+(e) + n \leq \mu(-\infty) \text{ for any } e \in E \text{ and} \\ \mu^+(V) + n \leq \ell\mu(-\infty) \end{array} \right\}.$$

Then

$$R = \bigoplus_{\mu \in t\mathcal{U}^{(0)}} \mathbb{K}T^{\mu} \quad \text{and} \quad \omega_R = \bigoplus_{\mu \in t\mathcal{U}^{(1)}} \mathbb{K}T^{\mu}.$$

Set

$$\mathfrak{p}_i = \mathbb{K}\{T^{\mu} \mid \mu \in t\mathcal{U}^{(0)}, \mu(v_i) > 0 \text{ or } \mu^+(V) < \ell\mu(-\infty)\}$$

and

$$\mathscr{P}_i = \{f \in \mathbb{R}^V \mid f(v_i) = 0 \text{ and } f^+(V) = \ell\}$$

for $0 \le i \le 2\ell$.

Then $\mathscr{P}_i$ is a face of STAB(G) corresponding to $\mathfrak{p}_i$, i.e., $E_{\mathbb{K}}[\mathscr{P}_i] = R/\mathfrak{p}_i$.

Lemma 16 $\dim \mathscr{P}_i = \ell$ *for any* i*.*

Theorem 17

$$\sqrt{\mathrm{tr}(\omega_R)} = \bigcap_{i=0}^{2\ell} \mathfrak{p}_i.$$

In particular, non-Gorenstein locus of R is a closed subset of $\mathrm{Spec}R$ of dimension $\ell + 1$.

Let $S = \bigoplus_{n\geq 0} S_n$ be a Cohen-Macaulay graded ring,

ω_S the graded canonical module of S

and $a(S)$ the a-invariant of S.

If there is an exact sequence

$$0 \to S \to \omega_S(-a) \to M \to 0,$$

with $M = 0$ or M is an Ulrich module, i.e., $e(M) = \mu(M)$, then we say that S is an almost Gorenstein ring.

Remark 18 $e(M) \geq \mu(M)$ in general.

Theorem 19 R is almost Gorenstein.

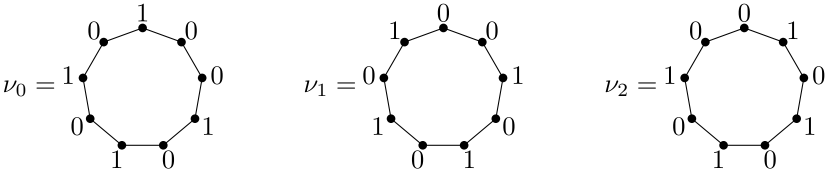
We define $\nu_i \in \mathbb{R}^V$ by

$$\nu_i(v_j) = \begin{cases} 1, & j - i \equiv 0, 2, 4, \ldots, 2\ell - 2 \pmod{2\ell + 1}, \\ 0, & \text{otherwise} \end{cases}$$

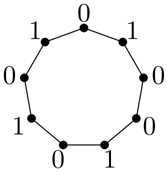
and $\mu \in \mathbb{R}^{V^-}$ by

$$\mu_i(x) = \begin{cases} \nu_i(x), & x \in V, \\ 1, & x = -\infty \end{cases}$$

for $0 \le i \le 2\ell$.



$\cdots$

$\nu_8 =$ 

Lemma 20 $\mu_i \in t\mathcal{U}^{(0)}$ for $0 \leq i \leq 2\ell$ and $T^{\mu_0}, \ldots, T^{\mu_{2\ell}}$ are algebraically independent.

Set $R^{(0)} = \mathbb{K}[T^{\mu_0}, \ldots, T^{\mu_{2\ell}}]$ and define $\eta_k \in \mathbb{Z}^{V^-}$ by

$$\eta_k(x) = \begin{cases} k, & x \in V, \\ 2k+1, & x = -\infty, \end{cases}$$

for $1 \leq k \leq \ell - 1$.

Remark 21 $\eta_k \in t\mathcal{U}^{(1)}$ and T^{η_1} is an element of ω_R with minimum degree. In particular, $a(R) = -3$.

Lemma 22 Let $\varphi\colon R \to \omega_R(3)$ be the R-linear map with $\varphi(1) = T^{\eta_1}$. Then

$$\mathrm{Cok}\varphi = \bigoplus_{k=2}^{\ell-1} R^{(0)} T^{\eta_k}$$

Corollary 23

$$e(\mathrm{Cok}\varphi) = \mu(\mathrm{Cok}\varphi).$$

Definition 24 An h-vector $(h_0, h_1, \ldots, h_s)$, $h_s \neq 0$ of a Cohen-Macaulay standard graded ring is called flawless if

(1) $h_i \leq h_{s-i}$ for $0 \leq i \leq \lfloor s/2 \rfloor$ and

(2) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor s/2 \rfloor}$.

Hibi conjectured in 1989 that any Cohen-Macaulay standard graded domain has a flawless h-vector. Niesi-Robbiano constructed a Cohen-Macaulay standard graded domain whose h-vector is $(1, 3, 5, 4, 4, 1)$, a counter example of Hibi's conjecture. Hibi-Tsuchiya computed the h-vector of the Ehrhart rings of the cycle graphs of length up to 11 and disproved Hibi's conjecture again. They also made the following Conjecture 25.

Let $(h_0, h_1, \ldots, h_s)$, $h_s \neq 0$ be the h-vector of R.

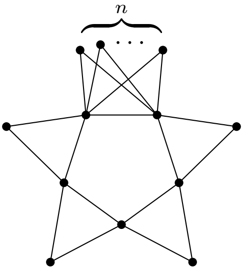
Then $s = \dim R + a(R) = 2\ell + 2 - 3 = 2\ell - 1$.

Conjecture 25 (Hibi-Tsuchiya) $h_s = 1$, $h_{s-1} = h_1$ and $h_{s-i} = h_i + (-1)^i$ for $2 \leq i \leq \lfloor s/2 \rfloor$.

Theorem 26 Conjecture 25 is true.

As a corollary of Theorem 26, we see the following.

Corollary 27 There is an infinite sequence of standard graded Cohen-Macaulay domains whose h-vectors are not flawless.



$a = -4,$

$n = 1,\ s = 7,\ h_4 = h_3 - 1,$

$n = 3,\ s = 9,\ h_5 = h_4 - 2,$

$n = 5,\ s = 11,\ h_6 = h_5 - 15,$

$n = 7,\ s = 13,\ h_7 = h_6 - 154,$

$n = 9,\ s = 15,\ h_8 = h_7 + 5670,$