# LEVELNESS VERSUS NEARLY GORENSTEINNESS OF HOMOGENEOUS DOMAINS

SORA MIYASHITA

ABSTRACT. Levelness and nearly Gorensteinness are well-studied properties of graded rings as a generalized notion of Gorensteinness. In this presentation, we compare the strength of these properties. We show that for any homogeneous affine semigroup ring, if it is nearly Gor with CM-type and projective dimension both 2, then it is level and its Hilbert series has nice form.

## 1. PRELIMINARIES

Let $\Bbbk$ be a field, and let $R$ be an $\mathbb{N}$-graded $\Bbbk$-algebra with a unique graded maximal ideal $\mathbf{m}$. We will always assume that $R$ is CM and admits a canonical module $\omega_R$.

- For a graded $R$-module $M$, we use the following notation:
  — Fix an integer $k$. Let $M(-k)$ denote the $R$-module whose grading is given by $M(-k)_n = M_{n-k}$ for any $n \in \mathbb{Z}$.
  —
$$\mathrm{tr}_R(M) = \sum_{\phi \in \mathrm{Hom}_R(M,R)} \phi(M)$$
  is called trace ideal of $R$. When there is no risk of confusion about the ring we simply write $\mathrm{tr}(M)$.
- Let $r(R)$ be CM-type.

Let us recall the definitions and facts of the nearly Gorensteinness and levelness of graded rings.

**Definition 1.1** (see [Sta, Chapter III, Proposition 3.2]). $R$ is *level* $\Leftrightarrow$ all the degrees of the minimal generators of $\omega_R$ are the same.

**Definition 1.2** (see [HHS, Definition 2.2]). $R$ is *nearly Gorenstein* $\Leftrightarrow$ $\mathrm{tr}(\omega_R) \supseteq \mathbf{m}$.
In particular, $R$ is Gor $\Leftrightarrow$ $\mathrm{tr}(\omega_R) = R$.

We recall some definitions about affine semigroups.

**Definition 1.3.** An *affine semigroup* $S$ is a fin.gen. sub-semigroup of $\mathbb{N}^d$.
$S$ is *homogeneous* $\Leftrightarrow$ all its minimal generators lie on an affine hyperplane not including origin. $\Leftrightarrow$ the affine semigroup ring $\Bbbk[S]$ is standard graded by assigning degree one to all the monomials corresponding to the minimal generators of $S$. In that case, we also say that $\Bbbk[S]$ is *homogeneous.*

**Theorem 1.4** (see [HHS, Corollary 3.5]). *Let* $S = \Bbbk[x_1, \cdots, x_n]$ *be a polynomial ring, let* $\mathbf{n} = (x_1, \cdots, x_n)R$ *and let*

$$\mathbb{F} : 0 \to F_p \xrightarrow{A} F_{p-1} \to \cdots \to F_1 \to F_0 \to R \to 0$$

*be a graded minimal free $S$-resolution of the CM ring $R = S/J$ with $J \subseteq \mathbf{n}^2$. Let $I_1(A)$ be an ideal of $R$ generated by all components of representation matrix of $A$. If $r(R) = 2$ and $\dim R > 0$, then $I_1(\phi_p) = \mathbf{n} \Leftrightarrow R$ is nearly Gor.*

## 2. Examples: nearly Gorensteinness versus levelness

We consider the following question.

**Question 2.1.** *If $R$ is nearly Gor, then is $R$ level?*

The following example show that in the case of non-domains, nearly Gorensteinness does not imply levelness.

**Example 2.2.** Let $S = \mathbb{Q}[x, y, z]$ be a graded polynomial ring with $\deg x = \deg y = \deg z = 1$. Consider a homogeneous ideal $I = (xz, yz, y^3)$ and define $R = S/I$, then the graded minimal free resolution of $R$ is as follows.

$$0 \to S(-3) \oplus S(-4) \xrightarrow{\begin{bmatrix} -y & 0 \\ x & -y^2 \\ 0 & z \end{bmatrix}} S(-2)^{\oplus 2} \oplus S(-3) \to S \to R \to 0.$$

Thus $r(R) = 2$ and $R$ is not level, and $R$ is CM because $\dim R = \operatorname{depth} R = 1 > 0$. Then, $R$ is nearly Gor by Theorem 1.4.

Even if $R$ is a domain, if it is not standard graded, we can find an example of nearly Gorenstein but not level.

**Example 2.3** (see [HHS, Remark 6.2])**.** Consider numerical semigroup ring $R = \Bbbk[t^5, t^6, t^7]$, then $R$ is not level but nearly Gor by Theorem 1.4.

Next, we consider the case of the CM standard graded domain. Surprisingly, even in that case, we can find the following example.

**Example 2.4.** Let $S = \mathbb{Q}[x, y, z]$ be a graded polynomial ring with $\deg x = \deg y = \deg z = 1$. Consider a homogeneous prime ideal $P = (x^3 - y^2 z, x^2 y - xyz - z^3, y^3 - xyz - xz^2 - z^3)$ and define $R = S/P$, then the graded minimal free resolution of $R$ is as follows.

$$0 \to S(-4) \oplus S(-5) \xrightarrow{\begin{bmatrix} -y & yz + z^2 \\ x + z & -y^2 - z^2 \\ -z & x^2 - xz + z^2 \end{bmatrix}} S(-3)^{\oplus 3} \to S \to R \to 0.$$

Thus $R$ is not level but nearly Gor by Theorem 1.4.

We reached the following Question.

**Question 2.5.** *Let $R = \Bbbk[S]$ be a CM homog affine semigroup ring. If $R$ is nearly Gor, then is $R$ level?*

We state the necessary results about the minimal free resolution of the codimension 2 lattice ideal based on [PS].

**Definition 2.6.** Let $S = \Bbbk[x_1, \cdots, x_n]$ be a polynomial ring and let $L$ be any sublattice of $\mathbb{Z}$. We put $\mathbf{x}^{\mathbf{a}} := {x_1}^{a_1} {x_2}^{a_2} \cdots {x_n}^{a_n}$ where $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$. Then its associated *lattice ideal* in $S$ is

$$I_L := (\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \ ; \ \mathbf{a}, \mathbf{b} \in \mathbb{N}^n \ \ \textit{and} \ \ \mathbf{a} - \mathbf{b} \in L).$$

Prime lattice ideals are called *toric ideals*. Prime binomial ideals and toric ideals are identical.

**Proposition 2.7** (see [PS, Comments 5.9 (a) and Theorem 6.1 (ii)]). *Let $S = \Bbbk[x_1, \cdots, x_n]$ be a polynomial ring. If $I$ is a codimension 2 lattice ideal of $S$ and the number of minimal generators of $I$ is 3, then $R = S/I$ is CM and the graded minimal free resolution of $R$ is the following form.*

$$0 \to S^2 \xrightarrow{\begin{bmatrix} u_1 & u_4 \\ u_2 & u_5 \\ u_3 & u_6 \end{bmatrix}} S^3 \to S \to R \to 0,$$

*where $u_i$ is a monomial of $S$ for all $1 \le i \le 6$.*

Note that a codimension 2 prime binomial ideal $I$ is CM but not Gor if and only if the number of minimal generators of $I$ is 3 (see [PS, Remark 5.8 and Theorem 6.1]). We denote $|\mathbf{a}|= \sum_{k=1}^{n} a_i$ where $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$.

**Theorem 2.8.** *Let $d \geqq 2$ and let $R$ be a $d$-dimensional homog affine semigroup. If $R$ is nearly Gor and $pd(R) = r(R) = 2$ , then $R$ is level and*

$$H(R, t) = \frac{1 + 2\sum_{i=1}^{s} t^i}{(1-t)^d}.$$

*Proof.* By the assumption, there exists a codimension 2 homogeneous prime binomial ideal $I$ such that $I$ is minimally generated by three elements and $R \cong S/I$, where $S = \Bbbk[x_1, \cdots, x_n]$ is a polynomial ring. Since $I$ is a codimension 2 lattice ideal and the number of minimal generators of $I$ is 3, $R$ is a $(n-2)$-dimensional CM ring and the graded minimal free resolution of $R$ is of the following form by Proposition 2.7.

$$0 \to S^2 \xrightarrow{A=\begin{bmatrix} u_1 & -u_4 \\ -u_2 & u_5 \\ u_3 & -u_6 \end{bmatrix}} S^3 \to S \to R \to 0.$$

Here, $u_i$ is a monomial of $S$ for all $1 \le i \le 6$. By using Hilbert-Burch Theorem (see [BH, Theorem 1.4.17]), $I$ is minimally generated by $f_1 = u_1u_5 - u_2u_4, f_2 = u_3u_4 - u_1u_6$ and $f_3 = u_2u_6 - u_3u_5$. Since $I$ is a prime binomial ideal, for all $i = 1, 2, 3$, there exist $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{N}^n$ such that $f_i = \mathbf{x}^{\mathbf{a}_i} - \mathbf{x}^{\mathbf{b}_i}$, $|\mathbf{a}_i|= |\mathbf{b}_i|$ and $\gcd(\mathbf{x}^{\mathbf{a}_i}, \mathbf{x}^{\mathbf{b}_i}) = 1$. We assume that $R$ is nearly Gor and show that $R$ is level.

- If $d = 2$, since $R$ is nearly Gor, $A$ may be assumed to have one of the following forms by Theorem 2.9.

  $$\text{(i) } A = \begin{bmatrix} x_1 & -x_4 \\ -x_2 & u_5 \\ x_3 & -u_6 \end{bmatrix} \text{ or (ii) } A = \begin{bmatrix} x_1 & -x_3 \\ -u_2 & x_4 \\ x_2 & -u_6 \end{bmatrix} \text{ or (iii) } A = \begin{bmatrix} x_1 & -x_3 \\ -x_2 & x_4 \\ u_3 & -u_6 \end{bmatrix}.$$

  (For example, there is also a possibility that $A = \begin{bmatrix} u_1 & -x_2 \\ -u_2 & x_3 \\ x_1 & -x_4 \end{bmatrix}$, but this can be regarded to be the same as (i).)

  In the cases of (i) and (ii), we see that all components of the matrix $A$ are variables $x_i$. Then the graded minimal free resolution of $R$ is as follows.

  $$0 \to S(-3)^{\oplus 2} \to S(-2)^{\oplus 3} \to S \to R \to 0.$$

Thus $R$ is level and $H(R,t) = H(S,t) - 3H(S(-2),t) + 2H(S(-3),t) = \dfrac{1+2t}{(1-t)^2}$.

(iii) Assume the case of $A = \begin{bmatrix} x_1 & -x_3 \\ -x_2 & x_4 \\ u_3 & -u_6 \end{bmatrix}$. Then the graded minimal free resolution of $R$ is as follows.

$$0 \to S(-N-2)^{\oplus 2} \xrightarrow{\begin{bmatrix} x_1 & -x_3 \\ -x_2 & x_4 \\ {x_3}^{n_1}{x_4}^{m_1} & -{x_1}^{n_2}{x_2}^{m_2} \end{bmatrix}} S(-N-1)^{\oplus 2} \oplus S(-2) \to S \to R \to 0,$$

where $N = \deg u_3$ Thus $R$ is level and $H(R,t) = \dfrac{1-(2t^{N+1}+t^2)+2t^{N+2}}{(1-t)^4} = \dfrac{1+2\sum_{i=1}^{N} t^i}{(1-t)^2}$.

- If $d = 3$ or 4, we see that all components of the matrix $A$ are variables $x_i$. Then it is clear that $R$ is level and $H(R,t) = \dfrac{1+2t}{(1-t)^2}$.
- If $d \geq 5$, $R$ cannot be nearly Gor by Theorem 1.4.

□

For homogeneous affine semigroup rings with type 3 or more, nearly Gorensteinness does not imply levelness in general.

**Theorem 2.9.** *For every $3 \leq d \leq 5$, there exists type $d$ non-level nearly Gor homog affine semigroup ring $R_d$.*

*Proof.* The following example exist.

- $R_3 = \Bbbk[s, st^2, st^4, st^5, st^7, st^9, st^{12}, st^{17}]$
- $R_4 = \Bbbk[s, st^4, st^9, st^{12}, st^{13}, st^{21}]$
- $R_5 = \Bbbk[s, st^6, st^7, st^9, st^{13}, st^{15}, st^{19}]$

□

For general homogeneous $d$-dimensional affine semigroup ring $R$, nearly Gorensteinness does not imply the equation $H(R,t) = \dfrac{1+r(R)\sum_{i=1}^{s} t^i}{(1-t)^d}$. Indeed, there are many counterexamples of $\mathrm{pd}(R) \geqq 4$.

**Example 2.10.** $R = \Bbbk[s, st^2, st^6, st^8, st^{11}, st^{17}, st^{23}]$ is nearly Gor and

$$H(R,t) = \frac{1+5t+9t^2+6t^3+2t^4}{(1-t)^2}.$$

However, for 2-dimensional homogeneous affine semigroup ring $R$ with $\mathrm{pd}(R) = 3$, the following example exists.

**Example 2.11.** $R = \Bbbk[s, st^{99}, st^{101}, st^{200}, st^{301}]$ is nearly Gor and

$$H(R,t) = \frac{1+3\sum_{i=1}^{100} t^i}{(1-t)^2}.$$

## References

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(S. Miyashita) Department of Pure And Applied Mathematics , Graduate School Of Information Science And Technology, Osaka University, Suita, Osaka 565-0871, Japan

*Email address*: miyashita.sora@ist.osaka-u.ac.jp