Conic divisorial ideals of toric rings and applications to stable set rings

Koji Matsushita (Osaka University)

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This talk is based on

K. Matsushita, Conic divisorial ideals of toric rings and applications to Hibi rings and stable set rings, arXiv:2210.02031.

Purposes of this talk ; we introduce

- an idea to determine a region representing conic divisorial ideals in the divisor class group of a toric ring.
- a description of the conic divisorial ideals of stable set rings of perfect graphs.

Setting

Let

- $\Bbbk$ be a field and $[n] = \{1, \dots, n\}$ for $n \in \mathbb{Z}_{>0}$.
- $\mathcal{V} = \{v_1, \dots, v_n\} \subset \mathbb{Z}^d$, where each v_i is primitive.
- $\tau = \mathrm{Cone}(\mathcal{V}) = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n$.
- $\tau^\vee = \{\mathbf{x} \in \mathbb{R}^d : \sigma_i(\mathbf{x}) \geq 0 \text{ for } i \in [n]\}$, where $\sigma_i(-) = \langle -, v_i \rangle$.

We assume that

"$\dim \tau = \dim \tau^\vee = d$" and "$v_i$'s are minimal generators of τ".

Definition

We define the toric ring

$$R = \Bbbk[\tau^\vee \cap \mathbb{Z}^d] = \Bbbk[t_1^{\alpha_1} \cdots t_d^{\alpha_d} : (\alpha_1, \dots, \alpha_d) \in \tau^\vee \cap \mathbb{Z}^d].$$

- R is a d-dimensional Cohen-Macaulay normal domain.

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we set

- $\mathbb{T}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Z}^d : \sigma_i(\mathbf{x}) \geq a_i \text{ for } i \in [n]\}$.
- $T(\mathbf{a})$; the module of R generated by all monomials whose exponent vector is in $\mathbb{T}(\mathbf{a})$.

Remark

- For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, we have $T(\mathbf{a}) = T(\ulcorner \mathbf{a} \urcorner)$, where $\ulcorner\ \urcorner$ means the round up and $\ulcorner \mathbf{a} \urcorner = (\ulcorner a_1 \urcorner, \cdots, \ulcorner a_n \urcorner)$.
- The module $T(\mathbf{a})$ is a divisorial ideal (rank one reflexive module) and any divisorial ideal of R takes this form. Therefore, we can identify each $\mathbf{a} \in \mathbb{Z}^n$ with the divisorial ideal $T(\mathbf{a})$.
- The isomorphic classes of divisorial ideals of R one-to-one correspond to the elements of the divisor class group $\mathrm{Cl}(R)$ of R.
- For $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}^n$, $T(\mathbf{a}) \cong T(\mathbf{a}') \Leftrightarrow \exists \mathbf{y} \in \mathbb{Z}^d$ s.t. $a_i = a'_i + \sigma_i(\mathbf{y})$ for all $i \in [n]$. Thus, we have $\mathrm{Cl}(R) \cong \mathbb{Z}^n / \sigma(\mathbb{Z}^d)$, where $\sigma = (\sigma_1, \dots, \sigma_n)$.

Definition (Bruns-Gubeladze (2003))

For $\mathbf{a} \in \mathbb{Z}^d$, we say that a divisorial ideal $T(\mathbf{a})$ is conic if there exists $\mathbf{x} \in \mathbb{R}^d$ such that $\mathbf{a} = \ulcorner \sigma(\mathbf{x}) \urcorner$.

Remark

- Up to isomorphism the conic divisorial ideals of R are exactly the direct summands of $R^{1/k}$ for $k \gg 0$, where $R^{1/k} = \Bbbk[\tau^\vee \cap (1/k\mathbb{Z})^d]$ is regarded as an R-module (Smith-Van den Bergh (1997), Bruns-Gubeladze (2003)).
- Since $R^{1/k}$ is a maximal Cohen-Macaulay (MCM) R-module, conic divisorial ideals of R are also MCM R-modules.
- For $k \gg 0$, $\mathrm{End}_R(R^{1/k})$ is a non-commutative resolution (NCR) of R (Špenko-Van den Bergh (2017), Faber-Muller-Smith (2019)).
- The endomorphism ring of the direct sum of some conic modules of R may be a non-commutative crepant resolution (NCCR).
  $\rightarrow$ NCCRs constructed in this way are called toric NCCRs.

- It is important to classify MCM divisorial ideals (including conic ones) of certain class of toric rings.

Example

It is known that

- a classification of MCM divisorial ideals is given in the case of toric rings whose divisor class group are $\mathbb{Z}$ or $\mathbb{Z}^2$ (Stanley (1982), Van den Bergh (1993)).
- a description of conic divisorial ideals is also given in the case of
  - Hibi rings (Higashitani-Nakajima (2019)).
    $\rightarrow$ this result is correct, but its proof is insufficient.
  - edge rings of complete multipartite graphs (Higashitani-M. (2022)).
    $\rightarrow$ not completely determined.

We give an idea to determine a region representing conic classes in $\mathrm{Cl}(R)$.

Description of conic divisorial ideals of toric rings

In what follows,

- we assume that $\mathrm{Cl}(R) \cong \mathbb{Z}^r$ for some $r \in \mathbb{Z}_{>0}$.
- we fix an isomorphism $\phi : \mathbb{Z}^n/\sigma(\mathbb{Z}^d) \to \mathbb{Z}^r$.
- for $i \in [n]$, let $\beta_i = \phi \circ \pi(\mathbf{e}_i)$, where $\mathbf{e}_i$ is the i-th basic vector in $\mathbb{Z}^n$ and $\pi : \mathbb{Z}^n \to \mathbb{Z}^n/\sigma(\mathbb{Z}^d)$ is the natural epimorphism.

We define

$$\mathcal{W}(R) = \left\{ \sum_{i=1}^{n} a_i \beta_i \in \mathbb{R}^r : a_i \in [0,1) \right\}.$$

Proposition (Špenko-Van den Bergh (2017))

Each element of $\mathcal{W}(R) \cap \mathbb{Z}^r$ one-to-one corresponds to a conic divisorial ideal of R.

Example

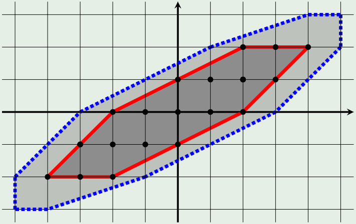
Let

$$\tau = \mathrm{Cone}(\{(1,0,0),(0,1,0),(1,-1,1),(-1,1,1),(-1,-1,3)\}).$$

Then, the divisor class group of the toric ring $R = \Bbbk[\tau^\vee \cap \mathbb{Z}^3]$ is isomorphic to $\mathbb{Z}^2$. We can compute the weights:

$$\beta_1 = (4,2),\ \beta_2 = (-2,-2),\ \beta_3 = (-3,-1),\ \beta_4 = (0,1),\ \beta_5 = (1,0).$$

$$\mathcal{W}(R) \cap \mathbb{Z}^2 = \left\{ (z_1, z_2) \in \mathbb{Z}^2 : \begin{array}{l} -2 \leq z_2 \leq 2 \\ -2 \leq z_1 - z_2 \leq 2 \\ -2 \leq z_1 - 2z_2 \leq 2 \end{array} \right\}$$



We want to determine the facet defining inequalities of a convex polytope representing conic classes.

We consider the lattice polytope

$$\mathcal{W}'(R) = \left\{ \sum_{i=1}^{n} a_i\beta_i \in \mathbb{R}^r : a_i \in [0,1] \right\}.$$

This is the Minkowski sum of lattice segments $\{a_i\beta_i : a_i \in [0,1]\}$, and hence $\mathcal{W}'(R)$ is a zonotope.

Proposition (BLSWZ "Oriented Matroids" (1999))

If there exist $\mathbf{n} \in \mathbb{Z}^r \setminus \{0\}$ and $\beta_{i_1}, \dots, \beta_{i_{r-1}}$ such that $\beta_{i_1}, \dots, \beta_{i_{r-1}}$ are linearly independent and $\langle \mathbf{n}, \beta_{i_j} \rangle = 0$ for all $j \in [r-1]$, then

$$F = \left\{ \sum_{\langle \mathbf{n}, \beta_i \rangle > 0} \beta_i + \sum_{\langle \mathbf{n}, \beta_i \rangle = 0} a_i\beta_i \in \mathbb{R}^r : a_i \in [0,1] \right\}$$

is a facet of $\mathcal{W}'(R)$. Conversely, all facets of $\mathcal{W}'(R)$ are obtained in this way.

Key Lemma

We have $\mathrm{int}(\mathcal{W}'(R)) = \mathcal{W}(R)$.

Therefore, if we have

$$\mathcal{W}'(R) = \left\{(z_1, \dots, z_r) \in \mathbb{R}^r : -q_i - 1 \le \sum_{j=1}^{r} c_{ij} z_j \le p_i + 1 \text{ for all } i \in [m]\right\}$$

for some positive integers m, p_i, q_i and some integers c_{ij}'s, where the greatest common divisor of $c_{i1}, \dots, c_{ir}$ is equal to 1 for all $i \in [m]$, then we can get the desired representation:

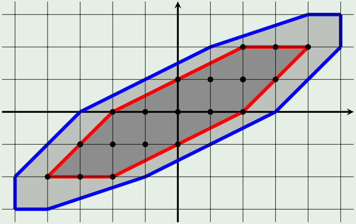
$$\mathcal{W}(R) \cap \mathbb{Z}^r = \left\{(z_1, \dots, z_r) \in \mathbb{Z}^r : -q_i \le \sum_{j=1}^{r} c_{ij} z_j \le p_i \text{ for all } i \in [m]\right\}.$$

- The equality $\mathrm{int}(\mathcal{W}'(R)) = \mathcal{W}(R)$ holds since the weights satisfy

$$\mathbb{Z}_{\ge 0}\beta_1 + \cdots + \mathbb{Z}_{\ge 0}\beta_n = \mathbb{Z}^r.$$

Example

$$\mathcal{W}'(R) = \left\{ (z_1, z_2) \in \mathbb{R}^2 : \begin{array}{l} -3 \le z_2 \le 3 \\ -3 \le z_1 - z_2 \le 3 \\ -3 \le z_1 - 2z_2 \le 3 \\ -5 \le z_1 \le 5 \\ -5 \le z_1 - 3z_2 \le 5 \end{array} \right\}.$$



Therefore, we have

$$\mathcal{W}(R) \cap \mathbb{Z}^2 = \left\{ (z_1, z_2) \in \mathbb{R}^2 : \begin{array}{l} -2 \le z_2 \le 2 \\ -2 \le z_1 - z_2 \le 2 \\ -2 \le z_1 - 2z_2 \le 2 \\ -4 \le z_1 \le 4 \\ -4 \le z_1 - 3z_2 \le 4 \end{array} \right\}.$$

By using this lemma, we can give a description of the conic divisorial ideals of

- Hibi rings (we can re-prove the result).
- stable set rings of perfect graphs.

Stable set rings

For a simple graph G, let

- $V(G) = \{1, \ldots, d\}$ denote the vertex set of G,
- $E(G)$ denote the edge set of G.

Definition

We say that $S \subset V(G)$ is a stable set (resp. a clique) if $\{v, w\} \notin E(G)$ (resp. $\{v, w\} \in E(G)$) for any distinct vertices $v, w \in S$. Note that the empty set and each singleton are regarded as stable sets.

Given a subset $W \subset V(G)$, let $\rho(W) = \sum_{i \in W} \mathbf{e}_i \in \mathbb{R}^d$, where we let $\rho(\emptyset)$ stands for the origin of $\mathbb{R}^d$.

Definition

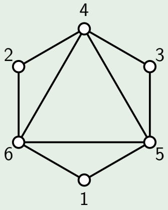
We define the stable set ring of G over $\Bbbk$ by setting

$$\Bbbk[\mathrm{Stab}_G] = \Bbbk[\ (\textstyle\prod_{i \in S} t_i) t_0 : S \text{ is a stable set of } G].$$

- The stable set ring of G can be described as the toric ring arising from a rational polyhedral cone if G is perfect.

Example

We consider the following perfect graph G:



We can see that G has maximal stable sets $\{1, 2, 3\}$, $\{1, 4\}$, $\{2, 5\}$ and $\{3, 6\}$. Thus, we have

$\Bbbk[\mathrm{Stab}_G] =$

$\Bbbk[t_0, t_1 t_0, t_2 t_0, \dots, t_6 t_0, t_1 t_2 t_0, t_1 t_3 t_0, t_2 t_3 t_0, t_1 t_2 t_3 t_0, t_1 t_4 t_0, t_2 t_5 t_0, t_3 t_6 t_0]$.

In what follows, we assume that G is a perfect graph with maximal cliques $Q_0, Q_1, \dots, Q_n$.

- In this case, we have $\mathrm{Cl}(\Bbbk[\mathrm{Stab}_G]) \cong \mathbb{Z}^n$ (Higashitani-M. (2022)).

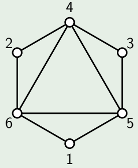
For $v \in V(G)$ and multisets $I, J \subset \{0, 1, \dots, n\}$, let

- $m_I(v) = |\{i \in I : v \in Q_i\}|$,
- $X_{IJ}^+ = \{v \in V(G) : m_I(v) - m_J(v) > 0\}$,
- $X_{IJ}^- = \{v \in V(G) : m_I(v) - m_J(v) < 0\}$.

Example

The graph G has 4 maximal cliques:

$$Q_0 = \{4, 5, 6\}, Q_1 = \{1, 5, 6\},$$
$$Q_2 = \{2, 4, 6\}, Q_3 = \{3, 4, 5\}.$$



Let $I = \{1, 2, 3\}$ and $J = \{0, 0, 0\}$. Then, we can see that

$$m_I(v) = \begin{cases} 1 & \text{if } v = 1, 2, 3, \\ 2 & \text{if } v = 4, 5, 6. \end{cases} \qquad m_J(v) = \begin{cases} 0 & \text{if } v = 1, 2, 3, \\ 3 & \text{if } v = 4, 5, 6. \end{cases}$$

and

$$X_{IJ}^+ = \{1, 2, 3\}, \quad X_{IJ}^- = \{4, 5, 6\}.$$

$$\mathcal{C}(G) = \Big\{ (z_1, \cdots, z_n) \in \mathbb{R}^n : -|J| + \sum_{v \in X_{IJ}^-} m_{IJ}(v) + 1 \le \sum_{i \in I} z_i - \sum_{j \in J} z_j \le |I| + \sum_{v \in X_{IJ}^+} m_{IJ}(v) - 1 \text{ for multisets } I, J \subset \{0, 1, \ldots, n\} \text{ with } |I| = |J| \text{ and } I \cap J = \emptyset \Big\},$$

where $m_{IJ}(v) = m_I(v) - m_J(v)$ and we let $z_0 = 0$.

(continued)

We can obtain the inequality

$$-|J| + \sum_{v \in X_{IJ}^-} m_{IJ}(v) + 1 = -5 \le z_1 + z_2 + z_3 \le 5 = |I| + \sum_{v \in X_{IJ}^+} m_{IJ}(v) - 1.$$

In this case, we have

$$\mathcal{C}(G) = \left\{ (z_1, z_2, z_3) \in \mathbb{R}^3 : \begin{array}{l} -1 \le z_1 \le 1 \\ -1 \le z_2 \le 1 \\ -1 \le z_3 \le 1 \end{array} \right\}.$$

Theorem (M. (2022))

Let G be a perfect graph with $n+1$ maximal cliques. Then, the conic divisorial ideals of $\Bbbk[\mathrm{Stab}_G]$ one-to-one correspond to the points in $\mathcal{C}(G) \cap \mathbb{Z}^n$.

Remark

- In the case of the Ehrhart ring of a chain polytope, which is the stable set ring of the comparability graph of a poset, we expect to be able to describe the conic class in terms of the poset.
- We construct a toric NCCR for a special family of stable set rings as an application of this theorem.