AN EXPLICIT CONSTRUCTION OF PERFECTOID ALMOST COHEN-MACAULAY ALGEBRAS IN MIXED CHARACTERISTIC

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Abstract. The existence of "well-behaved" non-Noetherian algebras over a Noetherian ring has many applications. Properties of such algebras are, for example, perfectoid and almost Cohen-Macaulay. In positive characteristic, the absolute integral closure and the perfect closure have these properties. There are some constructions of mixed characteristic analogues of these algebras.

In this talk, we look back at the previous constructions and construct a more explicit analogue of the perfect closure in mixed characteristic. This talk is based on joint work with Kazuma Shimomoto [IS].

1. Introduction

For a given Noetherian ring R, in particular complete Noetherian local domain, the existence of possibly non-Noetherian algebra over R and its properties play an important role in commutative ring theory. Properties of such algebras are, for example, perfectoid and almost Cohen-Macaulay. Recently, Bhatt and Scholze build a remarkable theory of perfectoidization by using prismatic cohomology in [BS22], which is a construction of perfectoid algebras. This theory constructs a mixed characteristic analogue of the perfect closure of positive characteristic (see [CLM$^+$22]).

However, perfectoidization is abstract. So, we construct a perfectoid almost Cohen-Macaulay algebra which is enough explicit and small by using an explicit representation of the perfect closure of positive characteristic.

In this talk (and report), first, we compare some constructions and their properties of positive characteristic and mixed characteristic, respectively. Second, we introduce our Main theorem, and finally, we talk about further conclusions after this talk.

In particular, we changed the title of the paper [IS] from *"An explicit construction of perfectoid almost Cohen-Macaulay algebras in mixed characteristic"* to *"A mixed characteristic analogue of the perfection of rings and its almost Cohen-Macaulay property"* to emphasize more that we focus on the similarity of the perfect closure.

2. Previous constructions

First, we show the previous constructions in positive characteristic and their analogues in mixed characteristic.

Notation 2.1. Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain with perfect residue field k whose characteristic is $p > 0$. Set $d \coloneqq \dim(R)$ and let $x_1, \dots, x_d, \dots, x_n$ be a system of generators of $\mathfrak{m}$ where $x_1, \dots, x_d$ is a system of parameters of R. By Cohen's structure theorem, there exists a complete regular local domain A and module-finite extension $A \hookrightarrow R$.

The contents of this section can be summarized as follows.

R : complete Noetherian local domain with perfect residue field k				
positive characteristic	referencces	properties	references	mixed characteristic
	[BMS18]	perfectoid	[BMS18]	
R^+	[HH91]	big CM	[Bha21]	$\widehat{R^+}$
	○	functorial	○	integral domain ([Hei22])
	×	explicit	×	
	[BMS18]	perfectoid	[BS22]	
	[RSS07]	almost CM	[CLM$^+$22]	
R_{perf}	○	functorial	×	$R^{A_{\infty,0}}_{\text{perfd}}$
	×	explicit	×	almost flat over A ([CLM$^+$22])
	$\subseteq$	$\widehat{R^+}$	$\overset{\exists!}{\longleftarrow}$	
	[BMS18]	perfectoid	[IS]	
	[RSS07]	almost CM	[IS]	
$R_\infty = \bigcup_{j \geq 0} R[x_1^{1/p^j}, \dots, x_n^{1/p^j}]$	○	functorial	×	$\widehat{\widehat{R}}_\infty$
	○	explicit	○	integral domain
	$\subseteq$	$\widehat{R^+}$	$\hookleftarrow$	almost flat over A ([IS])

2.1. **Absolute integral closure.** For R, one of the well-behaved non-Noetherian rings over R is the absolute integral closure R^+ of R, which is the integral closure of R in the algebraic closure of its fraction field.

2.1.1. *Positive characteristic case.* This R^+ is perfect by definition. Furthermore, we have the following. Note that the big Cohen-Macaulay property is a crucial point of a solution of direct summand conjecture [Hoc02].

Theorem 2.2 ([HH91])**.** *Let R be a positive characteristic. Then the absolute integral closure R^+ is a balanced big Cohen-Macaulay R-algebra. That is, every system of parameters of R is a regular sequence on R^+.*

2.1.2. *Mixed characteristic case.* We have a natural question, "How about the case of mixed characteristic?" The analogue of a perfect ring in mixed characteristic is a *perfectoid ring* in the sense of [BMS18]. Actually, perfectoid property is equivalent to perfect property in positive characteristic.

By definition, the p-adic completion $\widehat{R^+}$ of R^+ is perfectoid. Remark that, in positive characteristic, R^+ is already p-adically complete and thus $\widehat{R^+}$ is equal to R^+.

Furthermore, $\widehat{R^+}$ has some good properties. In [Hei22], this $\widehat{R^+}$ is also an integral domain. Not only the perfectoid property but also the big Cohen-Macaulay property is shown for $\widehat{R^+}$ by using the prismatic technique.

Theorem 2.3 ([Bha21]). *Let R be a mixed characteristic. Then the p-adic completion $\widehat{R^+}$ is a balanced big Cohen-Macaulay R-algebra.*

However, $\widehat{R^+}$ is so big and thus we want to take a much more explicit and smaller ring.

2.2. **perfect closure.**

2.2.1. *Positive characteristic case.* We back to the case of positive characteristic. We consider the perfect closure R_{perf} of R,

$$R_{\mathrm{perf}} \coloneqq \operatorname{colim}\{R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \cdots\} \cong \bigcup_{j\geq 0} R[x_1^{1/p^j}, \ldots, x_n^{1/p^j}] \subset R^+. \tag{2.1}$$

This has the same properties as above. In fact, R_{perf} is perfect by definition and almost Cohen-Macaulay. After this talk, we can get stronger property.

Theorem 2.4 ([RSS07, IS]). *The perfect closure R_{perf} is $(g)^{1/p^\infty}$-almost Cohen-Macaulay algebra with respect to $x_1, \ldots, x_d$ for some $g \in R_{\mathrm{perf}}$. Furthermore, R_{perf} is even $(g)^{1/p^\infty}$-almost flat and $(g)^{1/p^\infty}$-almost faithful A-algebra. That is, $\operatorname{Tor}_i^A(R_{\mathrm{perf}}, M)$ is $(g)^{1/p^\infty}$-almost zero for any A-module M and all $i > 0$, and $R_{\mathrm{perf}} \otimes_A M$ is not $(g)^{1/p^\infty}$-almost zero for any non-zero A-module M.*

Here, $(g)^{1/p^\infty}$-almost Cohen-Macaulay is defined as follows. In particular, almost flatness and almost faithfulness is followed by almost Cohen-Macaulayness.

Definition 2.5 ([Shi18]). For an R-algebra A and some element $\pi \in A$, we assume that A has a compatible sequence of p-power roots $\{\pi^{1/p^j}\}_{j\geq 0}$. Then an A-module M is called $(\pi)^{1/p^\infty}$-*almost Cohen-Macaulay algebra* with respect to $x_1, \ldots, x_d$ if the following are satisfied.

(1) For each $1 \leq k \leq d$, we have

$$(\pi)^{1/p^j} \cdot \frac{((x_1, \ldots, x_k)M :_M x_{k+1})}{(x_1, \ldots, x_k)M} = 0 \tag{2.2}$$

for all $j \geq 0$.

(2) There exists a $j \geq 0$ such that $\pi^{1/p^j} \cdot M/\mathfrak{m}M \neq 0$.

Almost Cohen-Macaulayness is an analogue of big Cohen-Macaulayness of almost mathematics. This is not a big Cohen-Macaulay algebra in general (see [RSS07]) but this property is useful in itself. If we want a big Cohen-Macaulay algebra over the almost Cohen-Macaulay algebra, we can take it by [And20, Die07].

2.2.2. *Mixed characteristic case.* In mixed characteristic, the same construction $\operatorname{colim}_{x \mapsto x^p} R$ is not a ring in general only a multiplicative monoid. Recalling that the perfectoid ring is an analogue of the perfect ring, one might think to use perfectoidization. By using perfectoidization, a mixed characteristic analogue of perfect closure of R has been constructed in [CLM+22].

Construction 2.6 ([CLM+22])**.** Set $x_1 = p$. Let $A \coloneqq W(k)[|t_2, \ldots, t_d|] \to R$ be a finite extension taken by Cohen's structure theorem, where t_i maps to x_i for $2 \leq i \leq d$. We define $A_{\infty,0}$ as the p-adic completion of

$$A^{nc}_{\infty,0} \coloneqq \bigcup_{j \geq 0} W(k)[p^{1/p^j}][|t_2^{1/p^j}, \ldots, t_d^{1/p^j}|]. \tag{2.3}$$

Then, we have a perfectoid ring $R^{A_{\infty,0}}_{\mathrm{perfd}} \coloneqq (A_{\infty,0} \otimes_A R)_{\mathrm{perfd}}$ by [BS22, Theorem 10.11]. This ring has some good properties arising from the almost purity theorem.

If R is positive characteristic $p > 0$, we take $A \coloneqq k[|t_1, \ldots, t_d|]$ and set $R^{A_{\infty,0}}_{\mathrm{perfd}}$ as the same construction. Then $R^{A_{\infty,0}}_{\mathrm{perfd}}$ is the same for the perfect closure of R (see [CLM+22, section 3]). This ring is used to define a mixed characteristic analogue of F-signature and Hilbert-Kunz multiplicity. However, $R^{A_{\infty,0}}_{\mathrm{perfd}}$ cannot be written explicitly yet.

So, we focus on the equality of (2.1). This representation in mixed characteristic is the main object in our main theorem in the next section. This construction is a generalization of the construction for regular local rings by [Shi16] and [And18, 3.4.5 (2)].

3. Main Theorem

3.1. **Main Theorem.** The main theorem is the following.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain of mixed characteristic $p > 0$ with perfect residue field k. Let $p, x_2, \ldots, x_n$ be a system of (not necessarily minimal) generators of the maximal ideal $\mathfrak{m}$ such that $p, x_2, \ldots, x_d$ forms a system of parameters of R. Choose compatible systems of p-power roots*

$$\{p^{1/p^j}\}_{j \geq 0}, \{x_2^{1/p^j}\}_{j \geq 0}, \ldots, \{x_n^{1/p^j}\}_{j \geq 0}$$

inside the absolute integral closure R^+. *Let* $\widetilde{R}_{\infty,\infty}$ *(resp.* $C(R_{\infty,\infty})$*) be the integral closure (resp. p-root closure) of*

$$R_{\infty,\infty} := \bigcup_{j\geq 0} R[p^{1/p^j}, x_2^{1/p^j}, \ldots, x_n^{1/p^j}]$$

in $R_{\infty,\infty}[1/p]$. *Let* $\widehat{\widetilde{R}}_{\infty,\infty}$, $\widehat{R}_{\infty,\infty}$, *and* $\widehat{C(R_{\infty,\infty})}$ *be the p-adic completions of* $\widetilde{R}_{\infty,\infty}$, $R_{\infty,\infty}$, *and* $C(R_{\infty,\infty})$ *respectively. Then there exists a nonzero element* $g \in \widehat{R}_{\infty,\infty}$ *and a compatible system of p-power roots* $\{g^{1/p^j}\}_{j\geq 0} \subseteq \widehat{R}_{\infty,\infty}$ *of g such that the following properties hold:*

(1) The ring map $\widehat{R}_{\infty,\infty} \to \widehat{\widetilde{R}}_{\infty,\infty}$ *is* $(p)^{1/p^\infty}$*-almost surjective.*
(2) $\widehat{\widetilde{R}}_{\infty,\infty}$ *is a perfectoid domain that is a subring of* $\widehat{R^+}$. *Moreover, the image of g under the map* $\widehat{R}_{\infty,\infty} \to \widehat{\widetilde{R}}_{\infty,\infty}$ *is a nonzero divisor.*
(3) Let $A := W(k)[|x_2, \ldots, x_d|] \hookrightarrow R$ *be a finite extension taken by Cohen's structure theorem. If R is a normal domain,* $\widehat{\widetilde{R}}_{\infty,\infty}$ *and* $\widehat{C(R_{\infty,\infty})}$ *are* $(pg)^{1/p^\infty}$*-almost flat and* $(pg)^{1/p^\infty}$*-almost faithful T-algebra.* [1]
(4) If R is a normal domain, for the above ring A and integral extension $A \to \widetilde{R}_{\infty,\infty}$, *there exists a nonzero element* $h \in A$ *such that* $A[1/h] \to \widetilde{R}_{\infty,\infty}[1/h]$ *is a filtered colimit of finite étale* $A[1/h]$*-algebras contained in* $\widetilde{R}_{\infty,\infty}[1/h]$.

3.2. **Method of the proof.** The important point of the theorem is that $\widehat{\widetilde{R}}_{\infty,\infty}$ is perfectoid and almost Cohen-Macaulay.

3.2.1. *perfectoid property.* The fact that $\widehat{\widetilde{R}}_{\infty,\infty}$ is perfectoid is deduced from the $(p)^{1/p^\infty}$-almost surjectivity of $C(R_{\infty,\infty}) \hookrightarrow \widetilde{R}_{\infty,\infty}$ and [Ces19, Proposition 2.1.8].

3.2.2. *almost Cohen-Macaulay property.* Being almost Cohen-Macaulay is proved by using tilting operation. We show that $\widehat{\widetilde{R}}^{\flat}_{\infty,\infty}$ is $(p^\flat)^{1/p^\infty}$-almost isomorphic to the perfect closure of some complete Noetherian local domain with perfect residue field. Then we use Theorem 2.4, André's flatness lemma, and André's Riemann extension theorem.

4. Future ideas

Finally, we recall some future ideas which were stated in this talk and introduce (partial) answers for these.

[1]When this talk is done, this (3) only states that $\widehat{\widetilde{R}}_{\infty,\infty}$ *is* $(pg)^{1/p^\infty}$*-almost Cohen-Macaulay algebra with respect to* $p, x_2, \ldots, x_d$. However, we can show a much stronger conclusion for this.

4.1. **The differences of perfectoidization.** The construction $R_{\mathrm{perfd}}^{A_{\infty,0}}$ and its properties are crucial points in the theory of F-signature and Hilbert-Kunz multiplicity in mixed characteristic.

Furthermore, by surjective map $\widehat{S}_\infty \twoheadrightarrow \widehat{R}_\infty$, we have a perfectoidization $(\widehat{R}_\infty)_{\mathrm{perfd}}$ as an honest perfectoid ring by [BS22, Theorem 7.4]. So we have perfectoid rings $R_{\mathrm{perfd}}^{A_{\infty,0}}$, $(\widehat{R}_\infty)_{\mathrm{perfd}}$, $\widehat{C(R_\infty)}$, and $\widehat{\widehat{R}}_\infty$ constructed by R. A natural question is the following.

Problem 1. *When are these perfectoid rings $R_{\mathrm{perfd}}^{A_{\infty,0}}$, $(\widehat{R}_\infty)_{\mathrm{perfd}}$, $\widehat{C(R_\infty)}$, and $\widehat{\widehat{R}}_\infty$ the (almost) isomorphic?*

The author's next work shows the following partial answer.

Proposition 4.1 ([Ish]). *A perfectoid ring $\widehat{\widehat{R}}_\infty$ is $(p)^{1/p^\infty}$-almost isomorphic to the perfectoidization $(\widehat{R}_\infty)_{\mathrm{perfd}}$ and its restriction to $\widehat{C(R_\infty)}$ induces an isomorphism of $\widehat{C(R_\infty)} \to (\widehat{R}_\infty)_{\mathrm{perfd}}$.*

4.2. **almost flatness.** In [CLM+22], one of the crucial properties of $R_{\mathrm{perfd}}^{A_{\infty,0}}$ is the (g)-almost flatness over A and in particular, (g)-almost p-complete flatness over $A_{\infty,0}$. So we expect that $\widehat{\widehat{R}}_{\infty,\infty}$ is $(pg)^{1/p^\infty}$-almost p-complete flatness for some regular ring. If $\widehat{\widehat{R}}_{\infty,\infty}$ holds this property, the theory of Hilbert-Kunz multiplicity proceeds in the same way as $R_{\mathrm{perfd}}^{A_{\infty,0}}$.

After this talk, we consider whether our construction holds this property. Considering the almost flatness and almost faithfulness of perfect closure in Theorem 2.4, we have the following as stated in the main theorem.

Proposition 4.2. *Let $A := W(k)[|x_2, \dots, x_d|] \hookrightarrow R$ be a finite extension taken by Cohen's structure theorem. If R is a normal domain, $\widehat{\widehat{R}}_{\infty,\infty}$ and $\widehat{C(R_{\infty,\infty})}$ are $(pg)^{1/p^\infty}$-almost flat and $(pg)^{1/p^\infty}$-almost faithful T-algebra.*

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