The canonical module and a Gorenstein criterion of a local log-regular ring

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This is not in a final form. The final version will be submitted to elsewhere for publication.

1 Log structure of commutative rings

In this report, all monoids and rings are commutative. We denote $\mathcal{Q}^*$ the set of units of a monoid $\mathcal{Q}$ and $\mathcal{Q}^{\mathrm{gp}}$ the group of the form of $a - b$ for all $a, b \in \mathcal{Q}$.

Definition 1.1. Let R be a ring, $\mathcal{Q}$ be a monoid and let $\alpha : \mathcal{Q} \to R$ be a monoid homomorphism. Then the triple $(R, \mathcal{Q}, \alpha)$ is called a *log ring*. Moreover if R is local and $\alpha^{-1}(R^\times) = \mathcal{Q}^*$ holds, $(R, \mathcal{Q}, \alpha)$ is called a *local log ring*.

In the following, we consider a monoid $\mathcal{Q}$ is a finitely generated submonoid of $\mathbb{N}^l$ for some $l > 0$ that satisfies the following condition: if $q \in \mathcal{Q}^{gp}$ such that $nq \in \mathcal{Q}$, then $q \in \mathcal{Q}$ (this condition is called *saturated*).

Next, we define the log-regularity of a commutative ring.

Definition 1.2. Let $(R, \mathcal{Q}, \alpha)$ be a local log ring, where R is Noetherian. Let I_α be the ideal of R generated by the image of $\mathcal{Q} \backslash \{0\}$ under α. Then we call the local log ring a *local log-regular ring* if it satisfies the following condition:

1. R/I_α is a regular local ring,
2. $\dim R = \dim R/I_\alpha + \dim \mathcal{Q}$, where $\dim \mathcal{Q}$ is the maximum length of the ascending chains of prime ideals of $\mathcal{Q}$.

We give an example of local log-regular rings.

Example 1.3. Let $\mathcal{Q}$ be the submonoid of $\mathbb{N}^2$ generated by $\langle (1,0), (1,1), (1,2), (1,3) \rangle$. Set $R := \mathbb{Z}_p[\![s, st, st^2, st^3]\!]$ and the monoid homomorphism $\alpha : \mathcal{Q} \to R$ which sends (i, j) to $s^i t^j$. Then the triple $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring.

A class of local log-regular rings has a structure theorem such as the Cohen's structure theorem.

Theorem 1.4 (Kato)**.** Let $(R, \mathcal{Q}, \alpha)$ be a local log ring, where R is Noetherian. Let k be a residue field of R. Then the following assertions hold.

1. Suppose that R is of equal characteristic. Then $(R, \mathcal{Q}, \alpha)$ is log-regular if and only if there exists a commutative diagram of the form

$$\begin{CD} \mathcal{Q} @>>> k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \\ @V{\alpha}VV @V{\cong}V{\phi}V \\ R @>>> \widehat{R}, \end{CD} \tag{1}$$

   where the top arrow is the natural injection and $\widehat{R}$ is the completion of R.

2. Suppose that R is of mixed characteristic. Let $C(k)$ be a Cohen ring of k and let $p > 0$ be the characteristic of k. Then $(R, \mathcal{Q}, \alpha)$ is log-regular if and only if there exists a commutative diagram

$$\begin{CD} Q @>>> C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \\ @V{\alpha}VV @VV{\phi}V \\ R @>>> \widehat{R}, \end{CD} \tag{2}$$

   where ϕ is a surjection whose kernel is a principal ideal generated by an element $\theta \in C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ whose constant term is p.

A class of local log-regular rings has many nice properties in commutative ring theory. Here are some of them.

Theorem 1.5. Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring.

1. (Kato) R is Cohen-Macaulay and normal.
2. (Gabber and Ramero) R is a splinter.
3. (Cai-Lee-Ma-Schwede-Tucker) Assume that R is complete. Then R is BCM-regular.

2 Main result

We give the main result of this report.

Theorem 2.1. Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring and let $x_1, \ldots, x_r$ be a sequence of elements of R such that $\overline{x_1}, \ldots, \overline{x_r}$ is a regular system of parameters for R/I_α. Then R has the canonical module and its form is

$$\langle (x_1 \cdots x_r)\alpha(x) \mid x \in \operatorname{relint} \mathcal{Q} \rangle, \tag{3}$$

where relint $\mathcal{Q}$ is the relative interior of $\mathcal{Q}$.

Remark 2.2. Professor Shunsuke Takagi pointed out to me that the ideal (3) in Theorem 2.1 is isomorphic to $\langle \alpha(x) \mid x \in \operatorname{relint} \mathcal{Q} \rangle$ because a canonical module of a local ring is unique up to isomorphism. Namely, the canonical module of a local log-regular ring has exactly the same description as a semigroup ring.

Example 2.3. Let $(R, \mathcal{Q}, \alpha)$ be the same as in Example 1.3. Since R/I_α is isomorphic to $\mathbb{Z}_p$, p is an element of R such that $\overline{p}$ is a regular system of parameters for R/I_α. Moreover, relint $\mathcal{Q}$ is generated by $(1, 1)$ and $(1, 2)$. Thus $\omega_R = \langle pst, pst^2 \rangle$.

From Theorem 2.1, we can obtain a Gorenstein criterion of a local log-regular ring.

Corollary 2.4. Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Then the following are equivalent:

1. R is Gorenstein,
2. $k[\mathcal{Q}]$ is Gorenstein for a fixed field k,
3. There exists an element $c \in \operatorname{relint} \mathcal{Q}$ such that $c + \mathcal{Q} = \operatorname{relint} \mathcal{Q}$.

By combining Corollary 2.4 with the classical result about the Gorenstein criterion of two-dimensional toric singularities, we can determine the structure of local log-regular rings whose associated monoidal dimension is two.

Proposition 2.5. Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring where the dimension of $\mathcal{Q}$ is two. Then R is Gorenstein if and only if $\mathcal{Q}$ is isomorphic to the submonoid of $\mathbb{N}^2$ generated by $(n + 1), (1, 1), (0, n + 1)$ for some $n \geq 1$.

Finally, we present the pseudo-rationality of a local log-regular ring. The proof is reduced to the complete case by the existence of a canonical module (namely, Theorem 2.1).

Proposition 2.6. Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Then R is pseudo-rational.

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