On finite generation of symbolic Rees rings

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Monday, November 14th, 2022
The 43rd Symposium on Commutative Algebra

Let A be a commutative ring and $\mathfrak{p}$ a prime ideal of A.

Definition (n-th symbolic power)

For any positive integer n, we put

$$\mathfrak{p}^{(n)} := \mathfrak{p}^n A_{\mathfrak{p}} \cap A$$

and call it the n-th symbolic power of $\mathfrak{p}$.

Definition (symbolic Rees ring)

We put

$$R_s(\mathfrak{p}) := A[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \ldots] \subset A[t]$$

and call it the symbolic Rees ring of A with respect to $\mathfrak{p}$.

Finite generation of the symbolic Rees ring is a very interesting and difficult problem.

Let K be a field.

(Assumption 1) : a, b, c are pairwise coprime positive integers such that $\sqrt{abc} \notin \mathbb{Q}$.

Suppose that $S = K[x, y, z]$ is a graded polynomial ring with $\deg(x) = a$, $\deg(y) = b$, $\deg(z) = c$.
Let $\mathfrak{p}$ be the kernel of the K-algebra map $\varphi \colon S = K[x, y, z] \to K[T]$ defined by $\varphi(x) = T^a$, $\varphi(y) = T^b$, $\varphi(z) = T^c$.

(Assumption 2) : $\mathfrak{p}$ is not complete intersection (i.e. $\mathfrak{p}$ is minimally generated by 3 elements).

Consider the symbolic Rees ring $R_s(\mathfrak{p})$.

Problem

Is $R_s(\mathfrak{p})$ Noetherian?

Finite generation of $R_s(\mathfrak{p})$ depends on a, b, c and $\mathrm{ch}(K)$.
There are many examples of finitely generated $R_s(\mathfrak{p})$.

Goto-Nishida-Watanabe (1994) : In the case of $\mathrm{ch}(K) = 0$, there are some examples of infinitely generated $R_s(\mathfrak{p})$.

Remark

In the case of $\mathrm{ch}(K) > 0$, we have no example of infinitely generated $R_s(\mathfrak{p})$.

Finite generation of $R_s(\mathfrak{p})$ is closely related to existence of the negative curve.

Definition (negative curve)

$f \in [\mathfrak{p}^{(r)}]_d$ is called a negative curve of $\mathfrak{p}$, if
(1) $d/r < \sqrt{abc}$, and
(2) f is an irreducible polynomial.

If there exists a negative curve of $\mathfrak{p}$, then it is uniquely determined.

Theorem (Cutkosky)

(1) If $R_s(\mathfrak{p})$ is Noetherian, then there exists a negative curve of $\mathfrak{p}$.
(2) In the case of $\mathrm{ch}(K) > 0$, $R_s(\mathfrak{p})$ is Noetherian if and only if there exists a negative curve of $\mathfrak{p}$.

Remark

We have no example where the negative curve of $\mathfrak{p}$ does not exist.

In the rest, we always assume the following three assumptions:

(Assumption 1) : a, b, c are pairwise coprime positive integers such that $\sqrt{abc} \notin \mathbb{Q}$.

(Assumption 2) : $\mathfrak{p}$ is not complete intersection (i.e. $\mathfrak{p}$ is minimally generated by 3 elements).

(Assumption 3) : A minimal generator of $\mathfrak{p}$ is the negative curve of $\mathfrak{p}$. ($\exists$ negative curve of $\mathfrak{p}$ with $r = 1$)

$S = K[x, y, z]$ is a graded polynomial ring with $\deg(x) = a$, $\deg(y) = b$, $\deg(z) = c$.

Then, we know

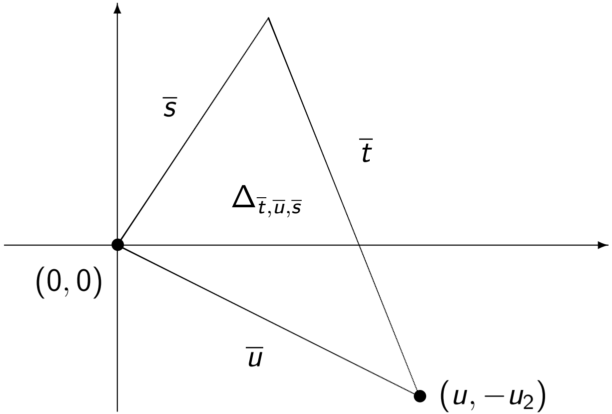
$$\mathfrak{p} = \mathrm{I}_2 \begin{pmatrix} x^{s_2} & y^{t_3} & z^{u_1} \\ y^{t_1} & z^{u_2} & x^{s_3} \end{pmatrix} = (x^s - y^{t_1} z^{u_1},\ y^t - z^{u_2} x^{s_2},\ z^u - x^{s_3} y^{t_3})$$

with positive integers $s_2, s_3, t_1, t_3, u_1, u_2$ such that $s = s_2 + s_3,\ t = t_1 + t_3,\ u = u_1 + u_2$, and moreover, we can prove $\gcd(s_2, s_3) = \gcd(t_1, t_3) = \gcd(u_1, u_2) = 1$.
Suppose that $z^u - x^{s_3} y^{t_3}$ is a negative curve of $\mathfrak{p}$, i.e., $uc < \sqrt{abc}$.
We put $\overline{t} = -t/t_3$, $\overline{u} = -u_2/u$, $\overline{s} = s_2/s_3$. Remark that

$$\overline{t} < -1 < \overline{u} < 0 < \overline{s}$$

is satisfied.

The triangle $\Delta_{\overline{t},\overline{u},\overline{s}}$:



Then, the Veronesean subring $S^{(ab)}$ of $S = K[x,y,z]$ is isomorphic to the Ehrhart ring of $\Delta_{\overline{t},\overline{u},\overline{s}}$.

We put $Q = (v-1, w-1)K[v^{\pm 1}, w^{\pm 1}]$.

$$
\begin{array}{ccccc}
S_{mab} & = & \displaystyle\bigoplus_{(\alpha,\beta)\in m\Delta_{\overline{t},\overline{u},\overline{s}}\cap\mathbb{Z}^2} Kv^\alpha w^\beta & \subset & K[v^{\pm 1}, w^{\pm 1}] \\
\cup & & \cup & & \\
[\mathfrak{p}^{(r)}]_{mab} & = & \left(\displaystyle\bigoplus_{(\alpha,\beta)\in m\Delta_{\overline{t},\overline{u},\overline{s}}\cap\mathbb{Z}^2} Kv^\alpha w^\beta\right) \cap Q^r & &
\end{array}
$$

In the case of $\mathrm{ch}(K) = 0$, for $n \in \mathbb{N}$ and $g = g(v, w) \in K[v^{\pm 1}, w^{\pm 1}]$,
$g \in Q^n \iff 0 \leqq \forall s + \forall t < n,\ \dfrac{\partial^{s+t} g}{\partial v^s \partial w^t}(1, 1) = 0.$

Definition (condition H_m)

For $m \in \mathbb{N}$, we say that the condition H_m is satisfied, if

$$\exists g \in [\mathfrak{p}^{(mu)}]_{mab} = \left(\bigoplus_{(\alpha,\beta)\in m\Delta_{\bar{t},\bar{u},\bar{s}}\cap\mathbb{Z}^2} K v^{\alpha} w^{\beta} \right) \cap Q^{mu} \quad \text{s.t.}$$

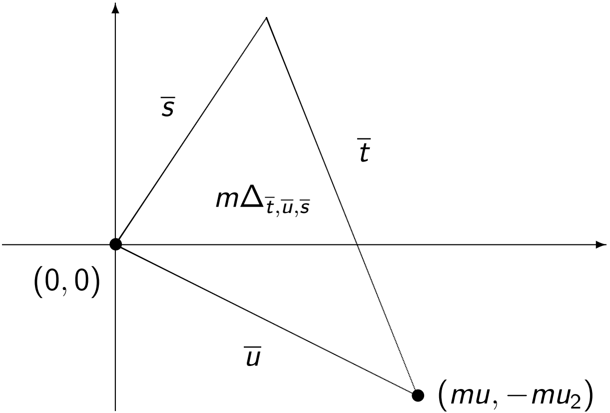
$$\text{" the constant term (or, the coefficient of } v^{mu}w^{-mu_2}\text{) of } g \text{ "} \neq 0.$$

Theorem (Huneke)

Assume (Assumption 1), (Assumption 2), (Assumption 3).
Then, $R_s(\mathfrak{p})$ is Noetherian if and only if $\exists m \in \mathbb{N}$ s.t. H_m is satisfied.

The condition H_m depends on $\mathrm{ch}(K)$.
$\mathrm{H}_m \Longrightarrow \mathrm{H}_{2m}, \mathrm{H}_{3m}, \mathrm{H}_{4m}, \ldots$ (In particular, $\mathrm{H}_1 \Longrightarrow \mathrm{H}_2, \mathrm{H}_3, \mathrm{H}_4, \ldots$).

The triangle $m\Delta_{\overline{t},\overline{u},\overline{s}}$:



For $i = 1, 2, \ldots, mu$, we put

$$\ell_i = \#\{(\alpha, \beta) \in m\Delta_{\overline{t},\overline{u},\overline{s}} \cap \mathbb{Z}^2 \mid \alpha = i\}.$$

Note that $\ell_{mu} = 1$ and $\ell_i \geqq 1$ for all $i = 1, 2, \ldots, mu$.

We sort the sequence $\ell_1, \ell_2, \ldots, \ell_{mu}$ into ascending order

$$\begin{array}{l} \ell'_1 \leqq \ell'_2 \leqq \cdots \leqq \ell'_{mu}. \\ \| \\ 1 \end{array}$$

Definition (condition EMU_m)

For $m \in \mathbb{N}$, we say that the condition EMU_m is satisfied, if

$$\ell'_i \geqq i \quad \text{for} \quad \forall i = 1, 2, \ldots, mu.$$

(EMU are the initials of Ebina, Matsuura, Uchisawa)

The condition EMU_m does not depend on $\mathrm{ch}(K)$.
$\mathrm{EMU}_{m+1} \Longrightarrow \mathrm{EMU}_m$.
In the case of $\mathrm{ch}(K) = 0$, $\mathrm{EMU}_m \Longrightarrow \mathrm{H}_m$.

Assume (Assumption 1), (Assumption 2), (Assumption 3).

Known facts

(1) In the case of $\mathrm{ch}(K) = p > 0$, $R_s(\mathfrak{p})$ is Noetherian. (Cutkosky, 1991) For $\mu \gg 0$, H_{p^μ} is satisfied.

Assume $\mathrm{ch}(K) = 0$.
(2) If $\exists m \in \mathbb{N}$ s.t. H_m is satisfied, then H_1 is satisfied. (Kurano-Nishida, 2019)
(3) Suppose $\ell_1 = 1$ or $\ell_{u-1} = 1$ for $\Delta_{\overline{t},\overline{u},\overline{s}}$ (In this case, EMU_1 is not satisfied. Remark $\ell_u = 1$ and $u = u_1 + u_2 \geqq 2$). Then, $R_s(\mathfrak{p})$ is not Noetherian. (González-Karu, 2016)
(4) Suppose $\ell_1 \geqq 3$ and $\ell_{u-1} \geqq 3$ for $\Delta_{\overline{t},\overline{u},\overline{s}}$. Then, EMU_m is satisfied for $\forall m \in \mathbb{N}$ (Therefore, $R_s(\mathfrak{p})$ is Noetherian).
(5) Suppose $\ell_{u-1} = n \geqq 3$, $\ell_1 = 2$, $\ell_2 = 3$, $\ldots$, $\ell_{n-1} = n$ and $n - 1 < u - 1$ for $\Delta_{\overline{t},\overline{u},\overline{s}}$ (In this case, EMU_1 is not satisfied). Then, $R_s(\mathfrak{p})$ is not Noetherian. (González-Karu, 2016)

Main theorem (arXiv:2204.01889, Theorem 1.2)

Assume (Assumption 1), (Assumption 2), (Assumption 3) and $\mathrm{ch}(K) = 0$.

Then, $R_s(\mathfrak{p})$ is Noetherian if and only if EMU_1 is satisfied.

1st step of proof

Suppose that $z^u - x^{s_3}y^{t_3}$ is a negative curve of $\mathfrak{p}$.
We classify $\Delta_{\overline{t},\overline{u},\overline{s}}$ for which EMU_1 is not satisfied.

$$\mathrm{EMU}_1 \text{ is not satisfied} \iff 1 \leqq \exists k < u \quad \text{s.t.}$$
$$\ell'_1 = 1,\ \ell'_2 = 2,\ \ldots,\ \ell'_{k-1} = k-1 \text{ and } \ell'_k = \ell'_{k+1} = k.$$
(k is called the minimal degree of $\Delta_{\overline{t},\overline{u},\overline{s}}$)

We may assume $\ell_1 \geqq \ell_{u-1}$ by exchanging x for y if necessary. We put

$$F := \{\Delta_{\overline{t},\overline{u},\overline{s}} \mid \ell_1 \geqq 3,\ \ell_{u-1} = 2,\ \mathrm{EMU}_1 \text{ is not satisfied}\}$$

and

$$F_{n,\lambda} := \{\Delta_{\overline{t},\overline{u},\overline{s}} \in F \mid \ell_1 = n,\ \text{min.deg.} = f_\lambda + f_{\lambda+1}\}$$

where $f_{-1} = 0,\ f_0 = 1,\ f_{\lambda+2} = (n-1)f_{\lambda+1} - f_\lambda$.
Then, $F = \coprod_{n\geqq 3,\, \lambda\geqq 0} F_{n,\lambda}$ holds.

2nd step

We prove

$$\forall n \geqq 3, \quad \forall \lambda \geqq 0, \quad \exists \Delta_{\overline{t},\overline{u},\overline{s}} \in F_{n,\lambda} \quad \text{s.t.} \quad R_s(\mathfrak{p}) \text{ is not Noetherian.}$$

3rd step

We prove

$$\forall n \geqq 3, \quad \forall \lambda \geqq 0, \quad \forall \Delta_{\overline{t},\overline{u},\overline{s}} \in F_{n,\lambda}, \quad R_s(\mathfrak{p}) \text{ is not Noetherian.}$$