# Reflexive modules over the endomorphism algebras of reflexive trace ideals

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# 1. Introduction

Let

- $R$ a commutative Noetherian ring with $(S_2)$ and $\mathrm{Q}(R)$ is Gorenstein
- $\operatorname{mod} R$ the category of finitely generated $R$-modules

For $M \in \operatorname{mod} R$,

$M$ is a reflexive $R$-module $\overset{def}{\iff}$ the natural map $M \to M^{**}$ is an isomorphism
$\iff$ $M_\mathfrak{p}$ is reflexive for $\mathfrak{p} \in \operatorname{Spec} R$ s.t. $\dim R_\mathfrak{p} = 1$ and $M$ satisfies $(S_2)$

where $(-)^* = \operatorname{Hom}_R(-, R)$ and

$$M \text{ satisfies } (S_2) \overset{def}{\iff} \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \inf\{2, \dim R_\mathfrak{p}\} \text{ for } \forall \mathfrak{p} \in \operatorname{Spec} R.$$

In what follows, let

- $(R, \mathfrak{m})$ a CM local ring with $\dim R = 1$, $\mathrm{Q}(R)$ is Gorenstein, and $|R/\mathfrak{m}| = \infty$
- $R \subseteq A \subseteq \mathrm{Q}(R)$ an intermediate ring s.t. $A \in \operatorname{mod} R$
- $\mathrm{CM}(A)$ the subcategory of $\operatorname{mod} A$ consisting of MCM $A$-modules
- $\mathrm{Ref}(A)$ the subcategory of $\operatorname{mod} A$ consisting of reflexive $A$-modules

For $M \in \operatorname{mod} A$,

$M$ is a MCM $A$-module $\overset{def}{\Longleftrightarrow}$ $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geq \dim A_\mathfrak{p}$ for $\forall \mathfrak{p} \in \operatorname{Spec} A$

$\Longleftrightarrow$ $M$ is a torsion-free $A$-module.

Then $\mathrm{Ref}(A) \subseteq \mathrm{CM}(A)$ and

$$\begin{aligned}\mathrm{Ref}(A) &= \{M \in \operatorname{mod} A \mid \exists\ 0 \to M \to F_0 \to F_1 \text{ s.t. } F_i \in \operatorname{mod} A \text{ is free}\}\\ &= \{M \in \operatorname{mod} A \mid \exists\ 0 \to M \to F \to X \to 0 \text{ s.t. } F \text{ is free, } X \in \mathrm{CM}(A)\}\\ &= \Omega\mathrm{CM}(A).\end{aligned}$$

Note that $\Omega\mathrm{CM}(A) = \mathrm{CM}(A) \iff A$ is a Gorenstein ring.

By setting $E = \mathrm{End}_R(\mathfrak{m}) \cong \mathfrak{m} : \mathfrak{m}$, we have

**Theorem 1.1 (Goto-Matsuoka-Phuong)**

$$\Omega\mathrm{CM}(E) = \mathrm{CM}(E) \iff R \text{ is almost Gorenstein and } \mathfrak{m} \text{ is stable.}$$

Recall that

- an ideal $I$ of $R$ is stable if $I^2 = aI$ for $\exists a \in I$
- $\mathfrak{m}$ is stable $\iff$ $R$ has minimal multiplicity
- $R$ is an almost Gorenstein ring if $\mathfrak{m}K \subseteq R$, where $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$.

Let $\Omega\mathrm{CM}'(R) = \{M \in \Omega\mathrm{CM}(R) \mid M \text{ doesn't have free summands}\}$.

**Theorem 1.2 (Kobayashi)**

(1) $\Omega\mathrm{CM}(E) \subseteq \Omega\mathrm{CM}'(R) \subseteq \mathrm{CM}(E)$.

(2) $\Omega\mathrm{CM}(E) = \Omega\mathrm{CM}'(R) \iff \mathfrak{m}$ *is stable.*

(3) $\Omega\mathrm{CM}'(R) = \mathrm{CM}(E) \iff R$ *is an almost Gorenstein ring.*

## Question 1.3

What happens if we take $\mathrm{End}_R(I)$?

Another motivation comes from the following.

## Theorem 1.4 (Dao-Iyama-Takahashi-Vial)

*Let $(A, \mathfrak{m})$ be an excellent henselian local normal domain with $\dim A = 2$ and $A/\mathfrak{m}$ is algebraically closed. Then*

$$A \text{ has a rational singularity} \iff \Omega\mathrm{CM}(A) \text{ is of finite type.}$$

A subcategory $\mathcal{X}$ of $\operatorname{mod} A$ is called of finite type if $\mathcal{X} = \mathrm{add}_A M$ for $\exists M \in \operatorname{mod} A$.

**Question 1.5**

When is $\Omega\mathrm{CM}(R)$ of finite type for a one-dimensional ring $R$?

Recall that

- $R$ is an almost Gorenstein ring $\iff$ $\Omega\mathrm{CM}'(R) = \mathrm{CM}(E)$
- $\Omega\mathrm{CM}'(R) = \{M \in \Omega\mathrm{CM}(R) \mid M$ doesn't have free summands$\}$.

**Corollary 1.6 (Kobayashi)**

*Suppose that $R$ is an almost Gorenstein ring. Then*

$$\Omega\mathrm{CM}(R) \text{ is of finite type} \iff \mathrm{CM}(E) \text{ is of finite type}$$

*where $E = \mathrm{End}_R(\mathfrak{m}) \cong \mathfrak{m} : \mathfrak{m}$.*

# 2. Main theorem

Note that $\mathfrak{m}$ is a regular reflexive trace ideal, once $R$ is not a DVR.

For an $R$-module $M$, consider the homomorphism

$$\tau : M^* \otimes_R M \to R, \ f \otimes m \mapsto f(m) \ \text{ for } f \in M^* \text{ and } m \in M$$

and set $\mathrm{tr}_R(M) = \mathrm{Im}\,\tau$.

We say that $I$ is a trace ideal of $R$ $\overset{def}{\iff}$ $I = \mathrm{tr}_R(M)$ for some $R$-module $M$
$\iff I = \mathrm{tr}_R(I)$
$\iff R : I = I : I.$ (when $I$ is regular)

- $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, if $R$ is not a DVR. (Goto-Matsuoka-Phoung)
- $M$ doesn't have free summands $\iff$ $\mathrm{tr}_R(M) \subseteq \mathfrak{m}$. (Lindo)
- $I = R : A$ is a regular reflexive trace ideal of $R$.

Hence $\Omega\mathrm{CM}'(R) = \{M \in \Omega\mathrm{CM}(R) \mid \mathrm{tr}_R(M) \subseteq \mathfrak{m}\}$.

An $\mathfrak{m}$-primary ideal $I$ of $R$ is called Ulrich, if $I$ is stable and $I/I^2$ is $R/I$-free.

For regular ideals in $R$, we have

$$\begin{array}{ccccccc} \text{Ulrich ideals} & \Longrightarrow & \text{good ideals} & \Longleftrightarrow & \text{stable trace ideals} & \Longrightarrow & \text{trace ideals} \\ & & \Downarrow & & & & \\ & & \text{reflexive ideals} & & & & \end{array}$$

If $R$ is Gorenstein, there are one-to-one correspondences for regular ideals: (Goto-Isobe-Kumashiro, Goto-Isobe-T)

- $\{$ trace ideals $\}$ $\longleftrightarrow$ $\{$ birational module-finite extensions $\}$
- $\{$ good ideals $\}$ $\longleftrightarrow$ $\{$ Gorenstein birational module-finite extensions $\}$
- $\{$ Ulrich ideals $\}$ $\longleftrightarrow$ $\{$ Gorenstein birational extensions $A$ s.t. $\mu_R(A) = 2$ $\}$
- $\{$ reflexive trace ideals $\}$ $\longleftrightarrow$ $\{$ reflexive birational module-finite extensions $\}$

Let $I$ be a regular reflexive trace ideal of $R$. We set

- $A = \mathrm{End}_R(I) \cong I : I$
- $\Omega\mathrm{CM}(R, I) = \{M \in \Omega\mathrm{CM}(R) \mid \mathrm{tr}_R(M) \subseteq I\}$.

Choose $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$. Set $S = R[K]$ and $\mathfrak{c} = R : S$.

**Theorem 2.1 (Main theorem)**

(1) $\Omega\mathrm{CM}(A) \subseteq \Omega\mathrm{CM}(R, I) \subseteq \mathrm{CM}(A)$.

(2) $\Omega\mathrm{CM}(A) = \Omega\mathrm{CM}(R, I) \iff I$ *is stable.*

(3) $\Omega\mathrm{CM}(R, I) = \mathrm{CM}(A) \iff IK = I \iff I \subseteq \mathfrak{c}$.

**Corollary 2.2**

$\Omega\mathrm{CM}(A) = \mathrm{CM}(A) \iff$ *$I$ is stable and $I \subseteq \mathfrak{c}$* $\iff$ *$A$ is a Gorenstein ring.*

In particular, since $\Omega\mathrm{CM}(R, \mathfrak{c}) = \mathrm{CM}(S)$, we have

$$\Omega\mathrm{CM}(S) = \Omega\mathrm{CM}(R, \mathfrak{c}) \iff S \text{ is a Gorenstein ring.}$$

For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, we denote by

- $\operatorname{ind} \mathcal{X}$ the set of isomorphism classes of indecomposable $R$-modules in $\mathcal{X}$.

## Corollary 2.3

*Let $R$ be a Gorenstein local domain with* $\dim R = 1$. *Then*

$$\begin{aligned} \operatorname{ind} \Omega\mathrm{CM}(R) &= \bigcup_{R \neq A \in \mathcal{Y}} \operatorname{ind} \mathrm{CM}(A) \cup \{[R]\} \\ &= \bigcup_{I \in \mathcal{T},\, I \neq R} \operatorname{ind} \mathrm{CM}(\mathrm{End}_R(I)) \cup \{[R]\} \end{aligned}$$

where

- $\mathcal{Y}$ is the set of birational module-finite extensions $A$ s.t. $A \in \mathrm{Ref}(R)$
- $\mathcal{T}$ is the set of regular reflexive trace ideals of $R$.

## Question 2.4

$\Omega\mathrm{CM}(R)$ is of finite type $\iff$ $\mathrm{CM}(A)$ is of finite type for some $A \in \mathcal{Y}$?

# 3. When is $\Omega\mathrm{CM}(R)$ of finite type?

Recall $R \subseteq K \subseteq \overline{R}$ s.t. $K \cong \mathrm{K}_R$, $S = R[K]$ and $\mathfrak{c} = R : S$. Then $S \in \mathcal{Y}$ and

- $R$ is a Gorenstein ring $\iff$ $R = K$ $\iff$ $R = S$ $\iff$ $R = \mathfrak{c}$
- $R$ is an almost Gorenstein ring $\iff$ $K/R \cong (R/\mathfrak{m})^{\oplus}$ $\iff$ $S/R \cong (R/\mathfrak{m})^{\oplus}$ $\iff$ $\mathfrak{m} \subseteq \mathfrak{c}$
- $R$ is an generalized Gorenstein ring if $R = \mathfrak{c}$, or $R \neq \mathfrak{c}$ and $K/R$ is $R/\mathfrak{c}$-free.

**Theorem 3.1**

*Suppose $R$ is a generalized Gorenstein ring with minimal multiplicity. Then*

$$|\operatorname{ind}\Omega\mathrm{CM}(R)| = \ell_R(R/\mathfrak{c}) + |\operatorname{ind}\mathrm{CM}(S)|.$$

*Hence, $\Omega\mathrm{CM}(R)$ is of finite type $\iff$ $\mathrm{CM}(S)$ is of finite type.*

**Corollary 3.2**

*Suppose $\mathrm{e}(R) = v(R) = 3$. Then $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \ell_R(R/\mathfrak{c}) + |\operatorname{ind}\mathrm{CM}(S)|$.*

## Corollary 3.3

*Suppose $R$ is a non-Gorenstein almost Gorenstein ring with minimal multiplicity. Then* $|\operatorname{ind}\Omega\mathrm{CM}(R)| = 1 + |\operatorname{ind}\mathrm{CM}(S)|$.

## Proposition 3.4

*Suppose $\overline{R}$ is a DVR, $\overline{R} \in \operatorname{mod} R$, and $\mathfrak{m}\overline{R} \subseteq R$. Then* $\operatorname{ind}\Omega\mathrm{CM}(R) = \{[R], [\overline{R}]\}$.

## Example 3.5

Let $A$ be a RLR with $n = \dim A \geq 2$. Let $X_1, X_2, \ldots, X_n$ be a regular sop of $A$ and set $P_i = (X_j \mid 1 \leq j \leq n,\ j \neq i)$ for $1 \leq i \leq n$. We set $R = A/\bigcap_{i=1}^{n} P_i$. Then $\operatorname{ind}\Omega\mathrm{CM}(R) = \{[R], [\overline{R}]\}$.

## Example 3.6

Suppose $\operatorname{ch} R > 0$. If $R$ is $F$-pure, then $\operatorname{ind}\Omega\mathrm{CM}(R) = \{[R], [\overline{R}]\}$, provided $\overline{R}$ is a DVR.

Note that

- if $R$ is a generalized Gorenstein ring with minimal multiplicity, then $S = R[K]$ is a Gorenstein ring.

## Corollary 3.7

*Let $R$ be the numerical semigroup ring over a field $k$. Suppose that $R$ is a generalized Gorenstein ring with minimal multiplicity. Then TFAE.*

(1) $\Omega\mathrm{CM}(R)$ *is of finite type.*

(2) $S = k[[H]]$ *is a semigroup ring of $H$, where $H$ is one of the following forms:*

  (a) $H = \mathbb{N}$,
  (b) $H = \langle 2, 2q+1 \rangle$ $(q \geq 1)$,
  (c) $H = \langle 3, 4 \rangle$, *or*
  (d) $H = \langle 3, 5 \rangle$.

Note that if $\mathrm{CM}(R)$ is of finite type, then

- $\mathcal{X}_R$ is a finite set (Goto-Ozeki-Takahashi-Watanabe-Yoshida)
- $R$ is analytically unramified (Krull, Leuschke-Wiegand)

where $\mathcal{X}_R$ denotes the set of Ulrich ideals of $R$.

### Theorem 3.8

*If $\Omega\mathrm{CM}(R)$ is of finite type, then $\mathcal{X}_R$ is finite and $R$ is analytically unramified.*

### Example 3.9

Let $(A, \mathfrak{m})$ be a CM local ring with $\dim A = 1$, $\exists\, \mathrm{K}_A$, $|A/\mathfrak{m}| = \infty$. Assume $\mathrm{Q}(A)$ is a Gorenstein ring. We set

$$R = A \ltimes A.$$

Then, because $|\mathcal{X}_R| = \infty$, we have $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \infty$.

We say that $R$ is an Arf ring, if every integrally closed regular ideal is stable.

## Theorem 3.10 (cf. Dao, Dao-Lindo, Isobe-Kumashiro)

*Suppose $\overline{R}$ is a local ring. If $R$ is an analytically unramified Arf ring, then $\Omega\mathrm{CM}(R)$ is of finite type. In particular, $\mathcal{X}_R$ is finite.*

## Example 3.11

Let $R = k[[t^3, t^4]]$. Then $|\operatorname{ind}\Omega\mathrm{CM}(R)| = |\operatorname{ind}\mathrm{CM}(R)| < \infty$, but $R$ is not an Arf ring.

## Example 3.12

Let $R = k[[t^3, t^7]]$. Then

$$|\mathcal{X}_R| = |\{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}| < \infty$$

provided $k$ is finite. However $|\operatorname{ind}\Omega\mathrm{CM}(R)| = \infty$ and $R$ is not an Arf ring.

**Thank you for your attention.**