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Abstracts

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On the Hilbert coefficients of graded modules over graded rings

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Let $R = \oplus_{n\in\mathbb{N}} R_n$ be a Noetherian $\mathbb{N}$-graded ring such that R_0 is an Artinian local ring and $R = R_0[R_1]$, where $\mathbb{N}$ denotes the set of non-negative integers. Moreover, let $M = \oplus_{n\in\mathbb{N}} M_n$ be a finitely generated $\mathbb{N}$-graded module over R. We set $s = \dim_R M$. Then there exist integers $\mathrm{e}_0(M), \mathrm{e}_1(M), \ldots, \mathrm{e}_s(M)$ such that

$$\sum_{i=0}^{n} \ell_{R_0}(\, M_i \,) \;=\; \sum_{i=0}^{s} (-1)^i \cdot \mathrm{e}_i(M) \cdot \binom{n+s-i}{s-i}$$

for $n \gg 0$. We call $\mathrm{e}_i(M)$ the i-th Hilbert coefficient of M. As is well known, if we choose $f_1, \ldots, f_s \in R_1$ so that they form an sop for M, then we have

$$\mathrm{e}_0(M) \;\leq\; \ell_R(\, M/(f_1, \ldots, f_s)M \,)$$

and the equality holds if and only if M is a Cohen-Macaulay R-module. The purpose of this talk is to give a generalization of this fact, which is a result on $\mathrm{e}_i(M)$ for any $i > 0$. We say that a sequence $f_1, f_2, \ldots, f_r$ of homogeneous elements of R is an M-filter-regular sequence if $a_i \notin P$ for any integer i with $1 \leq i \leq r$ and any $P \in \mathrm{Ass}_R\, M/(f_1, \ldots, f_{i-1})M$ with $R_1 \not\subseteq P$. The main result can be stated as follows.

Theorem *Suppose $s > 0$ and i is an integer with $0 \leq i < s$. Then if we choose $f_1, \ldots, f_{s-i} \in R_1$ so that they form an M-filter-regular sequence, we have*

$$\dim_R M/(f_1, \ldots, f_{s-i})M = i$$

and the following assertions hold.

(1) $\begin{cases} \mathrm{e}_i(M) \leq \mathrm{e}_i(M/(f_1, \ldots, f_{s-i})M) & \textit{if } i \textit{ is even,} \\ \mathrm{e}_i(M) \geq \mathrm{e}_i(M/(f_1, \ldots, f_{s-i})M) & \textit{if } i \textit{ is odd.} \end{cases}$

(2) $\mathrm{e}_i(M) = \mathrm{e}_i(M/(f_1, \ldots, f_{s-i})M)$ *if and only if* $\mathrm{depth}_R\, M \geq s - i$.

If $r \leq s$ and a sequence $f_1, \ldots, f_r$ of elements of R_1 is an M-filter-regular sequence, then it is an ssop for M. In this talk, we consider a sufficient condition for the converse of this implication to be true.

THE FIRST EULER CHARACTERISTIC AND THE DEPTH OF ASSOCIATED GRADED RINGS

KAZUHO OZEKI

The homological property of the associated graded ring of an ideal is an important problem in commutative algebra. In this talk, we explore the structure of the associated graded ring of $\mathfrak{m}$-primary ideals in the case where the first Euler characteristic attains almost minimal value in a Cohen-Macaulay local ring.

Throughout this talk, let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we may assume the residue class field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let

$$R = \mathrm{R}(I) := A[It] \quad \subseteq A[t] \text{ and } R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \subseteq A[t, t^{-1}]$$

denote, respectively, the Rees algebra and the extended Rees algebra of I. Let

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}$$

denotes the associated graded ring of I. Let $M = \mathfrak{m}G + G_+$ be the graded maximal ideal in G. Let $Q = (a_1, a_2, \cdots, a_d) \subseteq I$ be a parameter ideal in A which forms a reduction of I. Then, we set

$$\chi_1(a_1t, a_2t, \ldots, a_dt; G) := \ell(G/(a_1t, a_2t, \ldots, a_dt)G) - \mathrm{e}(a_1t, a_2t, \ldots, a_dt; G_M)$$

and call it the *first Euler characteristic* of G relative to $a_1t, a_2t, \ldots, a_dt$ (c.f.[1, 2]), where $\ell(*)$ and $\mathrm{e}(*)$ denote the length and the multiplicity symbol, respectively.

It is well-known that $\chi_1(a_1t, a_2t, \ldots, a_dt; G) \geq 0$ holds true, and the equality $\chi_1(a_1t, a_2t, \ldots, a_dt : G) = 0$ holds true if and only if the associated graded ring G is Cohen-Macaulay. The aim of this talk is to explore the structure of the associated graded ring G with $\chi_1(a_1t, a_2t, \ldots, a_dt; G) = 1$ and, in particular, we prove that $\mathrm{depth} G = d - 1$.

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GORENSTEINNESS FOR NORMAL TANGENT CONES OF GEOMETRIC IDEALS

TOMOHIRO OKUMA, KEI-ICHI WATANABE, KEN-ICHI YOSHIDA

Throughout this talk, let $(A, \mathfrak{m})$ be an excellent normal local domain containing an algebraically closed field. In this case, for any $\mathfrak{m}$-primary integrally closed ideal $I \subset A$, there exist a resolution of singularities $X \to \operatorname{Spec} A$ and an antinef cycle Z on X such that I can be represented as follows:

$$I\mathcal{O}_X = \mathcal{O}_X(-Z), \qquad I = H^0(X, \mathcal{O}_X(-Z))$$

Put $q(nI) = \dim_K H^1(X, \mathcal{O}_X(-nZ))$ for every integer $n \geq 1$. Then the normal reduction number $\bar{\mathrm{r}}(I)$ can be desribed in terms of $q(nI)$:

$$\begin{aligned}\bar{\mathrm{r}}(I) &= \min\{r \in \mathbb{Z}_{\geq 0} \,|\, \overline{I^{n+1}} = Q\overline{I^n}\ (n \geq r)\} \\ &= \min\{n \in \mathbb{Z}_{\geq 0} \,|\, q((n-1)I) = q(nI)\}.\end{aligned}$$

An $\mathfrak{m}$-primary integrally closed ideal I is called an *elliptic ideal* (resp. a *p_g-ideal*) if $\bar{\mathrm{r}}(I) = 2$ (resp. $\bar{\mathrm{r}}(I) = 1$). For such an ideal $I \subset A$, we consider the following geometric blow-up algebra:

$$\overline{G}(I) := \bigoplus_{n \geq 0} \overline{I^n}/\overline{I^{n+1}}.$$

We call this algebra the *normal tangent cone* of I.

It is well-known that $\overline{G}(I) = G(I)$ is Gorenstein if and only if I is good if A is Gorenstein and I is a p_g-ideal. Moreover, it is known that $\overline{G}(I)$ is Cohen-Macaulay for any elliptic ideal I. The main purpose of this talk is to prove the following theorem.

Theorem 1. *Suppose that A is Gorenstein and,let $I \subset A$ be an elliptic ideal. For any minimal reduction $Q \subset I$, the following conditions are equivalent*:

1. *$\overline{G}(I)$ is Gorenstein.*
2. *$Q \colon I = Q + \overline{I^2}$.*
3. *$\ell_A(\overline{I^2}/QI) = \ell_A(A/I)$.*
4. *$\bar{e}_2(I) = \ell_A(A/I)$, where $\bar{e}_2(I)$ is the second normal Hilbert coefficient of I.*
5. *$KZ = -Z^2$, where $K = K_X$ is the canonical divisor on X.*

Put $p_g(A) := \dim_K H^1(X, \mathcal{O}_X)$, the geometric genus of A.

Example 2. Let A be as above.

1. If $\bar{\mathrm{r}}(\mathfrak{m}) \leq 2$, then $\overline{G}(\mathfrak{m})$ is Gorenstein.
2. If $p_g(A) \leq 2$, then $\overline{G}(\mathfrak{m})$ is Gorenstein.
3. Let $a \geq 3$ be an integer. If we put $A = \mathbb{C}[[x, y, z]]/(x^a + y^{2a-1} + z^{2a-1})$, then $\overline{G}(\mathfrak{m})$ is not Gorenstein.

Homological dimension of tensor products of modules

Toshinori Kobayashi *

Throughout R denotes a commutative Noetherian ring, and all R-modules are assumed to be finitely generated. In this talk, we explain our result on homological dimension of a tensor product $M \otimes_R N$ of R-modules M and N. Our project was initiated by the discussions of the first coauthor with Roger Wiegand concerning the following questions:

Question 1. Let M and N be R-modules. If $\mathrm{pd}_R(M) < \infty$ and $\mathrm{pd}_R(N) < \infty$, then must $\mathrm{pd}_R(M \otimes_R N) < \infty$?

Question 2. If M and N are R-modules such that $\mathrm{pd}_R(M \otimes_R N) < \infty$, then must $\mathrm{pd}_R(M) < \infty$ or $\mathrm{pd}_R(N) < \infty$?

Wiegand proved:

Theorem 3 (Wiegand [2]). *Let R be a local ring. Then the following conditions are equivalent:*

(i) *If M is an R-module such that $\mathrm{pd}_R(M) < \infty$, then $\mathrm{pd}_R(M \otimes_R M) < \infty$.*

(ii) $\mathrm{depth}(R) = 0$ *or R is regular.*

One of our aims is to explain our construction of examples that give a negative answer to Question 2. On the other hand, we also plan to explain our observation that Question 2 is true under some conditions. In this direction, the following is one of our main results:

Theorem 4. *Let R be a commutative ring and let M and N be R-modules, where M is totally reflexive. If $\mathrm{pd}_R(M \otimes_R N) < \infty$, then M is projective and $\mathrm{pd}_R(N) < \infty$.*

This is joint work with Olgur Celikbas and Souvik Dey.

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NON-NORMAL EDGE RING SATISFYING (S_2)-CONDITION

NAYANA SHIBU DEEPTHI

Let G be a finite simple connected graph on the vertex set $V(G) = [d]$ and let $E(G) = \{e_1, \ldots, e_n\}$ be the edge set of G. Let us consider, $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ to be the polynomial ring in d variables over a field K. For an edge $e = \{i, j\}$ in $E(G)$, we define $\mathbf{t}^e := t_i t_j$. The subring of $K[\mathbf{t}]$ generated by $\mathbf{t}^{e_1}, \ldots, \mathbf{t}^{e_n}$ is called the edge ring of G, denoted by $K[G]$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the canonical unit coordinate vectors of $\mathbb{R}^d$ and for some $e = \{i, j\} \in E(G)$, we define $\rho(e) := \mathbf{e}_i + \mathbf{e}_j$. Let S_G be the affine semigroup generated by $\rho(e_1), \ldots, \rho(e_n)$. Then, the edge ring $K[G]$ is the affine semigroup ring of S_G.

The Cohen-Macaulayness of the edge ring $K[G]$ in terms of the corresponding graph G has been a subject of extensive research. Given that the edge ring $K[G]$ is an affine semigroup ring, it is known from [2, Theorem 1] that, if $K[G]$ is normal then $K[G]$ is Cohen-Macaulay.

Note that Serre's condition (S_2) is a necessary condition for $K[G]$ to be Cohen-Macaulay. Based on these insights, Higashitani and Kimura [1] have provided the necessary condition for an edge ring to satisfy (S_2)-condition.

This talk is based on the preprint [3] and the main theorem in this talk is as follows:

Theorem. *Given integers d and n such that, $d \geq 7$ and $d + 1 \leq n \leq \frac{d^2-7d+24}{2}$, there exists a finite simple connected graph G with $|V(G)| = d$ and $|E(G)| = n$ such that, the edge ring $K[G]$ is non-normal and satisfies (S_2)-condition.*

For that, we introduce the graph $G_{a,b}$, whose edge ring $K[G_{a,b}]$ is non-normal and further prove that the edge ring $K[G_{a,b}]$ satisfies (S_2)-condition. Then, we focus on the step wise removal of edges from $G_{a,b}$, such that each new graph obtained per step also satisfies both non-normality and (S_2)-condition. Moreover, we prove that any addition of new edges to the graph $G_{a,b}$, either affects the non-normality of the edge ring or leads to the violation of (S_2)-condition. Finally, we provide supporting evidences for our main theorem and list the conclusions.

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On finite generation of symbolic Rees rings

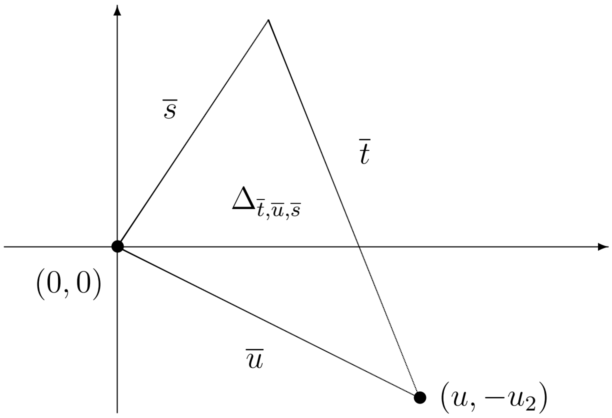
Taro Inagawa and Kazuhiko Kurano (Meiji University)

In this talk, we describe some necessary and sufficient condition for finite generation of symbolic Rees rings. Finite generation of the symbolic Rees ring is a very interesting and difficult problem. Furthermore, this talk is mainly based on Inagawa-Kurano [1].

Let K be a field. Suppose that a, b, c are pairwise coprime positive integers such that $\sqrt{abc} \notin \mathbb{Q}$. Let $\mathfrak{p}$ be the kernel of the K-algebra map $\varphi\colon P = K[x, y, z] \to K[T]$ defined by $\varphi(x) = T^a$, $\varphi(y) = T^b$, $\varphi(z) = T^c$. Assume that $\mathfrak{p}$ is not complete intersection (i.e. $\mathfrak{p}$ is minimally generated by 3 elements). Then, we know

$$\mathfrak{p} = (x^s - y^{t_1}z^{u_1},\ y^t - z^{u_2}x^{s_2},\ z^u - x^{s_3}y^{t_3})$$

with positive integers $s_2, s_3, t_1, t_3, u_1, u_2$ such that $s = s_2 + s_3$, $t = t_1 + t_3$, $u = u_1 + u_2$, and moreover, we can prove $\gcd(s_2, s_3) = \gcd(t_1, t_3) = \gcd(u_1, u_2) = 1$. We put $\overline{t} = -t/t_3$, $\overline{u} = -u_2/u$, $\overline{s} = s_2/s_3$. Remark $\overline{t} < -1 < \overline{u} < 0 < \overline{s}$. Here, consider the triangle $\Delta_{\overline{t},\overline{u},\overline{s}}$ as follows:



The slopes of edges of this triangle are $\overline{t}, \overline{u}, \overline{s}$ respectively.

Definition. For $i = 1, 2, \ldots, u$, we put

$$\ell_i = \#\{(\alpha, \beta) \in \Delta_{\overline{t},\overline{u},\overline{s}} \cap \mathbb{Z}^2 \mid \alpha = i\}.$$

Note that $\ell_u = 1$ and $\ell_i \geqq 1$ for all $i = 1, 2, \ldots, u$. We sort the sequence $\ell_1, \ell_2, \ldots, \ell_u$ into ascending order $\ell'_1 \leqq \ell'_2 \leqq \cdots \leqq \ell'_u$.

We say that the <u>condition EMU</u> is satisfied for (a, b, c) if $\ell'_i \geqq i$ for all $i = 1, 2, \ldots, u$.

We put $\mathfrak{p}^{(n)} = \mathfrak{p}^n P_{\mathfrak{p}} \cap P$, and call it the nth symbolic power of $\mathfrak{p}$. Consider the symbolic Rees ring $R_s(\mathfrak{p}) := P[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2, \mathfrak{p}^{(3)}t^3, \ldots] \subset P[t]$.

Theorem. *Let a, b, c be pairwise coprime positive integers such that $\sqrt{abc} \notin \mathbb{Q}$. Let K be a field of characteristic 0. Assume that $\mathfrak{p}$ is not complete intersection. Suppose that $z^u - x^{s_3}y^{t_3}$ is a negative curve, i.e., $\sqrt{abc} > uc$.*

Then, $R_s(\mathfrak{p})$ is Noetherian if and only if the condition EMU is satisfied for (a, b, c).

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STANLEY–REISNER RINGS WITH LOW CODIMENSION

NAOKI TERAI (OKAYAMA UNIVERSITY)

This is based on a joint work with M. R. Pournaki, M. Poursoltani, and S. Yassemi.

Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set $[n]$. Let S be a polynomial ring over a field k and I_Δ be the Stanley-Reisner ideal of Δ and $k[\Delta] = S/I_\Delta$ the Stnaley-Reisner ring of Δ.

A local cohomology is one of the most important tool in commutative ring theory. But in general it is not finitely generated. Then sometimes its dual is considered. In this article we focus on the dimension of the dual modules of local cohomology modules $K^j_{k[\Delta]} = \mathrm{Hom}_k(H^j_{\mathfrak{m}}(k[\Delta]), k)$ for $0 \le j \le d-1$. Set $d_i = \dim K^j_{k[\Delta]}$ for $0 \le j \le d-1$ and we call $(d_0, d_1, \ldots, d_d)$ the $K_{k[\Delta]}$-vector. This vector is important since it contains the information on (S_r) and CM_t property and the depth of the Stanley–Reisner ring $k[\Delta]$.

Proposition 0.1. *Let Δ be a $(d-1)$-dimensional pure simplicial complex such that I_Δ is of height $h \ge 2$ and let $(d_0, d_1, \ldots, d_{d-1}, d_d)$ be the $K_{k[\Delta]}$-vector. If $i \le d-h+1$, then*

$$d_i + 1 \le \max\left\{d_{i+1} - 1, \ldots, d_{i+h-1} - (h-1)\right\}.$$

Definition 0.2. Let Δ be a $(d-1)$-dimensional pure simplicial complex. Then for $d \ge r \ge 2$, the *S_r-depth* of $k[\Delta]$, denoted by S_r-depth $k[\Delta]$, is defined to be

$$\min\left\{j \mid \dim K^j_{k[\Delta]} \ge j - r + 1\right\}.$$

Definition 0.3. Let Δ be a $(d-1)$-dimensional pure simplicial complex. Then for $0 \le t \le d-1$, the *CM_t-depth* of $k[\Delta]$, denoted by CM_t-depth $k[\Delta]$, is defined to be

$$\min\{j \mid \dim K^j_{k[\Delta]} \ge t\}.$$

Theorem 0.4. *Let Δ be a $(d-1)$-dimensional pure simplicial complex on the vertex set $[n]$ such that I_Δ is of height h and $1 \le p \le d$. Then*

(1) *If $K^j_{k[\Delta]} = 0$ for $p-h+1 \le j \le p-1$, then* $\mathrm{depth}\, k[\Delta] \ge p$.

(2) *Let $r \ge 2$ be an integer. If $\dim K^j_{k[\Delta]} \le j - r$ for $p-h+1 \le j \le p-1$, then* $\mathrm{S}_r\mathrm{depth}\, k[\Delta] \ge p$.

(3) *Let $t \ge 0$ be an integer. If $\dim K^j_{k[\Delta]} \le t-1$ for $p-h+1 \le j \le p-1$, then* $\mathrm{CM}_t\mathrm{depth}\, k[\Delta] \ge p$.

Corollary 0.5. *Let Δ be a $(d-1)$-dimensional pure simplicial complex on the vertex set $[n]$ such that I_Δ is of height 2 and $1 \le p \le d$. Then*

(1) *If $K^{p-1}_{k[\Delta]} = 0$, then* $\mathrm{depth}\, k[\Delta] \ge p$.

(2) *If* $K^{d-1}_{k[\Delta]} = 0$*, then* $k[\Delta]$ *is Cohen-Macaulay.*

In [4] a $(d-1)$-dimensional non-Cohen-Macaulay Buchsbaum (or equivalently (S_{d-1})) complex with codimension two is characterized as the Alexander dual of $(d+2)$-gon.

In this article we classify (S_{d-2}), (S_{d-3}) comlexes with codimension two. Using these classifications we compute two invariants: the h-vector of $k[\Delta]$ and the arithmetical rank of I_Δ.

We give a complete characteriazation for d-dimensional Stanley-Reisner rings with Serre condition (S_{d-1}), (S_{d-2}) (S_{d-3}) of codimension two. Further we give a negative answer for the following question:

Question 0.6. [1, Question 2.6] Let d and r be integers with $d \geq r \geq 2$ and let $\mathfrak{h} = (h_0, h_1, \ldots, h_d)$ be the h-vector of a simplicial complex in such a way that the following conditions hold:

(1) $(h_0, h_1, \ldots, h_r)$ is an M-vector, and

(2) $\binom{i}{i}h_r + \binom{i+1}{i}h_{r+1} + \cdots + \binom{i+d-r}{i}h_d$ is nonnegative for every i with $0 \leq i \leq r \leq d$.

Does there exist a $(d-1)$-dimensional (S_r) simplicial complex Δ with $h(\Delta) = \mathfrak{h}$?

The *arithmetical rank* of an ideal I, denoted by $\mathrm{ara}I$, is defined by the minimal number r of elements $a_1, \ldots, a_r \in S$ such that

$$\sqrt{(a_1, \ldots, a_r)} = \sqrt{I}.$$

For a squarefree monomial ideal I Lyubeznik [3] proved that

$$\mathrm{pd}S/I \leq \mathrm{ara}I, \tag{1}$$

where $\mathrm{pd}S/I$ denotes the projective dimension of S/I. Under what condition does the equality $\mathrm{ara}I = \mathrm{pd}S/I$ hold? Kimura [2] showed that the equlity holds for the case that the Cohen-Macaulay Stanley-Reisner ideals of height two. How about non-Cohen-Maculay ideals of height two? We show the following result:

Theorem 0.7. *Let* Δ *be a* $(d-1)$*-dimensional pure non-Cohen-Macaulay simplicial complex on the vertex set* $[n]$ *such that* I_Δ *is of height two and* $n \geq 7$*. If* Δ *is* (S_{d-3})*, then* $\mathrm{ara}\, I_\Delta = \mathrm{pd}\, k[\Delta] = 3$.

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Levelness versus Nearly Gorensteinness of homogeneous domains

Sora Miyashita (Osaka University)*

Let R be a homogeneous ring with a unique graded maximal ideal $\mathbf{m}$. We will always assume that R is Cohen-Macaulay and admits a canonical module ω_R.

Definition 1. For a graded R-module M, let $\mathrm{tr}_R(M)$ be the sum of the ideals $\phi(M)$ with $\phi \in \mathrm{Hom}_R(M, R)$. Thus,

$$\mathrm{tr}_R(M) = \sum_{\phi \in \mathrm{Hom}_R(M,R)} \phi(M).$$

When there is no risk of confusion about the ring we simply write $\mathrm{tr}(M)$.

Definition 2 (see [2, Definition 2.2])**.** R is *nearly Gorenstein* if $\mathrm{tr}(\omega_R) \supseteq \mathbf{m}$. In particular, R is nearly Gorenstein but not Gorenstein if and only if $\mathrm{tr}(\omega_R) = \mathbf{m}$ (see [2, Lemma 2.1]).

Definition 3 (see [3, Chapter III, Proposition 3.2])**.** R is *level* if all the degrees of the minimal generators of ω_R are the same.

The following result is the main theorem of this talk.

Theorem 4. *(1) Nearly Gorensteinness does not necessarily imply the levelness of homogeneous domains.*

(2) Nearly Gorensteinness necessarily imply levelness of affine semigroup rings whose projective dimension and Cohen-Macaulay type are 2.

We also discuss Stanley-Reisner rings of low-dimensional simplicial complexes.

Theorem 5. *(a) Every 0-dimensional simplicial complex is nearly Gorenstein and level.*
(b) Let Δ be a 1-dimensional connected simplicial complex. The following conditions are equivalent.
(1) Δ is nearly Gorenstein;
(2) Δ is Gorenstein on the punctured spectrum $\mathrm{Spec}(R) \setminus \{\mathbf{m}\}$*;*
(3) Δ is locally Gorenstein (i.e., $\Bbbk[\mathrm{link}_\Delta(\{i\})]$ *is Gorentein for all vertex i);*
(4) Δ is a path or a cycle.
(c) Every 1-dimensional nearly Gorenstein simplicial complex is level.

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CONIC DIVISORIAL IDEALS OF TORIC RINGS AND APPLICATIONS TO STABLE SET RINGS

KOJI MATSUSHITA

This talk is based on [2]. Let $C \subset \mathbb{R}^d$ be a d-dimensional strongly convex rational polyhedral cone. We define the *toric ring* of C over a field $\Bbbk$ by setting

$$R = \Bbbk[C \cap \mathbb{Z}^d] = \Bbbk[t_1^{\alpha_1} \cdots t_d^{\alpha_d} : (\alpha_1, \cdots, \alpha_d) \in C \cap \mathbb{Z}^d].$$

Note that R is a d-dimensional Cohen-Macaulay normal domain. Recently, conic divisorial ideals, which are a certain class of divisorial ideals (rank one reflexive modules) and were introduced in [1], and their applications are well studied. For example, the endomorphism ring of the direct sum of some conic modules of R may be a non-commutative crepant resolution (NCCR) of R. In considering the construction of NCCRs and other applications, it is important to classify conic divisorial ideals of certain class of toric rings.

In this talk, we introduce an idea to determine a region representing conic classes in the divisor class group of R and a description of the conic divisorial ideals of stable set rings of perfect graphs.

Let G be a simple graph on the vertex set $V(G) = \{1, \ldots, d\}$ with the edge set $E(G)$. We say that $S \subset V(G)$ is a *stable set* (resp. a *clique*) if $\{v, w\} \notin E(G)$ (resp. $\{v, w\} \in E(G)$) for any distinct vertices $v, w \in S$. Note that the empty set and each singleton are regarded as stable sets.

We define the *stable set ring* of G over $\Bbbk$ by setting

$$\Bbbk[\mathrm{Stab}_G] = \Bbbk[\ (\textstyle\prod_{i \in S} t_i) t_0 : S \text{ is a stable set of } G].$$

The stable set ring of G can be described as the toric ring arising from a rational polyhedral cone if G is perfect. In what follows, we assume that G is a perfect graph with maximal cliques $Q_0, Q_1, \ldots, Q_n$.

For $v \in V(G)$ and a multiset $L \subset \{0, 1, \ldots, n\}$, let $m_L(v) = |\{l \in L : v \in Q_l\}|$. Moreover, for multisets $I, J \subset \{0, 1, \ldots, n\}$, we set

$$X_{IJ}^{+} = \{v \in V(G) : m_{IJ}(v) > 0\} \quad \text{and} \quad X_{IJ}^{-} = \{v \in V(G) : m_{IJ}(v) < 0\},$$

where $m_{IJ}(v) = m_I(v) - m_J(v)$. We define

$$\mathcal{C}(G) = \Bigg\{ (z_1, \cdots, z_n) \in \mathbb{R}^n : \\ -|J| + \sum_{v \in X_{IJ}^{-}} m_{IJ}(v) + 1 \le \sum_{i \in I} z_i - \sum_{j \in J} z_j \le |I| + \sum_{v \in X_{IJ}^{+}} m_{IJ}(v) - 1 \\ \text{for multisets } I, J \subset \{0, 1, \ldots, n\} \text{ with } |I| = |J| \text{ and } I \cap J = \emptyset \Bigg\},$$

where we let $z_0 = 0$. Note that an infinite number of inequalities appears in $\mathcal{C}(G)$, but in fact, only a finite number of inequalities are needed, and hence $\mathcal{C}(G)$ is a convex polytope.

Theorem. *The conic divisorial ideals of* $\Bbbk[\mathrm{Stab}_G]$ *one-to-one correspond to the points in* $\mathcal{C}(G) \cap \mathbb{Z}^n$.

In [2], we construct an NCCR for a special family of stable set rings as an application of this theorem. If time permits, we will introduce it too.

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REVISITING KRULL'S THEOREM

KAZUFUMI ETO, JUN HORIUCHI, AND KAZUMA SHIMOMOTO

1. Introduction

This talk is based on the joint work with K. Shimomoto and K. Eto [4]. The complete version of this resarch will be submitted to elsewhere. In this talk, we revisit the Krull's theorem. Krull proved the following result in [3]. We reprove this theorem by using properties of complete integral closedness of valuation rings.

Proposition 1.1 (Krull)**.** *We set V be a valuation ring and its field of fractions K.*

(1) Suppose that V is one-dimensional ring, then V is completely integrally closed in K.

(2) Suppose that V has a height-one prime ideal, then the complete integral closure of V in K is a valuation ring of rank one. If V does not have a height-one prime ideal, then the complete integral closure of V is K.

As an application, we have the following. See also [1, Proposition 6.2].

Proposition 1.2. *We set V be a one-dimensional valuation ring and let $t \in V$ be any element that is neither zero nor a unit. Then the t-adic completion $\widehat{V}$ of V is a valuation ring of dimension one. In addition, the natural map $V \to \widehat{V}$ is injective.*

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The canonical module and a Gorenstein criterion of a local log-regular ring

Shinnosuke Ishiro (Nihon University)

Let R be a commutative ring, let $\mathcal{Q}$ be a monoid, and let $\alpha : \mathcal{Q} \to R$ be a homomorphism of monoids. Then we call the triple $(R, \mathcal{Q}, \alpha)$ a *log ring*. A log ring $(R, \mathcal{Q}, \alpha)$ is *local* if R is local and $\alpha^{-1}(R^\times) = \mathcal{Q}^*$, where $\mathcal{Q}^*$ (resp. $R^\times$) is the set of units of $\mathcal{Q}$ (resp. R).

A monoid $\mathcal{Q}$ is called *fine* if it is integral and finitely generated. An integral monoid $\mathcal{Q}$ is called *saturated* if the following condition holds: for any $x \in \mathcal{Q}^{gp}$, if $nx \in \mathcal{Q}$ for some $n \geq 1$, then $x \in \mathcal{Q}$.

Here we give the definition of local log-regular rings.

Definition 1 *Let $(R, \mathcal{Q}, \alpha)$ be a local log ring, where R is Noetherian and $\overline{\mathcal{Q}} = \mathcal{Q}/\mathcal{Q}^*$ is fine and saturated. Let I_α be the ideal of R generated by $\alpha(\mathcal{Q}^+)$, where $\mathcal{Q}^+$ is the set of non-units of $\mathcal{Q}$. Then $(R, \mathcal{Q}, \alpha)$ is called a local log-regular ring if it satisfies the following conditions:*

1. *R/I_α is a regular local ring.*
2. *$\dim R = \dim R/I_\alpha + \dim \mathcal{Q}$.*

A class of local log-regular rings is introduced by Kazuya Kato in the study of logarithmic geometry ([2]). One of the reasons of an importance of this class is that this has the structure theorem like Cohen's structure theorem. From this structure theorem, one can deduce that the underlying ring R of local log-regular ring $(R, \mathcal{Q}, \alpha)$ is isomorphic to the completion of a monoid algebra over a fields (if R is of equal characteristic) or to some quotient ring of the completion of a monoid algebra over a complete discrete valuation ring (if R is of mixed characteristic). From the view of the theorem, we expect that local log-regular rings have similar properties of affine normal semigroup rings. The following theorem is derived from this perspective.

Theorem 2 ([1]) *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring where $\mathcal{Q} \subset \mathbb{N}^n$ for some $n \geq 0$. Let $x_1, \ldots, x_r$ be a sequence of elements of R such that $\overline{x_1}, \ldots, \overline{x_r}$ is a regular system of parameters on $R/I_{\mathcal{Q}}$.*

1. *The ideal $\langle (x_1 \cdots x_r)\alpha(a) \mid a \in \operatorname{relint} \mathcal{Q} \rangle$ is the canonical module of R, where $\operatorname{relint} \mathcal{Q}$ is the relative interior of $\mathcal{Q}$.*
2. *R is Gorenstein if and only if there exists an element $c \in \mathcal{Q}$ such that $c + \mathcal{Q} = \operatorname{relint} \mathcal{Q}$.*

In this talk, we will discuss about this theorem, including a comparison with the results for semigroup rings.

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An explicit construction of perfectoid almost Cohen-Macaulay algebras in mixed characteristic

Ryo Ishizuka*

This talk is based on joint work with Kazuma Shimomoto [IS].

The existence of "well-behaved" non-Noetherian algebras over a Noetherian ring R of mixed characteristic $(0, p)$ has many applications. Furthermore, how such algebras exist is also an interesting problem in itself.

However, these algebras, which are often perfectoid (almost) Cohen-Macaulay, have a complicated structure. In [IS], by using some perfectoid techniques, we construct perfectoid almost Cohen-Macaulay algebras as follows.

Main Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a complete local domain of mixed characteristic $p > 0$ with perfect residue field k and let $p, x_2, \ldots, x_n$ be a system of (not necessarily minimal) generators of the maximal ideal $\mathfrak{m}$ such that $p, x_2, \ldots, x_d$ forms a system of parameters of R. Choose compatible systems of p-power roots $\{p^{1/p^j}\}_{j\geq 0}, \{x_2^{1/p^j}\}_{j\geq 0}, \ldots, \{x_n^{1/p^j}\}_{j\geq 0}$ inside the absolute integral closure R^+. Let $\widetilde{R}_{\infty,\infty}$ be the integral closure of*

$$R_{\infty,\infty} := \bigcup_{j\geq 0} R[p^{1/p^j}, x_2^{1/p^j}, \ldots, x_n^{1/p^j}] \tag{1}$$

in $R_{\infty,\infty}[1/p]$ and Let $\widehat{\widetilde{R}}_{\infty,\infty}$ and $\widehat{R}_{\infty,\infty}$ be the p-adic completions of $\widetilde{R}_{\infty,\infty}$ and $R_{\infty,\infty}$, respectively. Then there exists a nonzero element $g \in \widehat{R}_{\infty,\infty}$ and a compatible system of p-power roots $\{g^{1/p^j}\}_{j\geq 0} \subseteq \widehat{R}_{\infty,\infty}$ of g such that the following properties hold:

1. *The ring map $\widehat{R}_{\infty,\infty} \to \widehat{\widetilde{R}}_{\infty,\infty}$ is $(p)^{1/p^\infty}$-almost surjective.*
2. *$\widehat{\widetilde{R}}_{\infty,\infty}$ is a perfectoid domain that is a subring of $\widehat{R^+}$. Moreover, the image of g under the map $\widehat{R}_{\infty,\infty} \to \widehat{\widetilde{R}}_{\infty,\infty}$ is a nonzero divisor.*
3. *$\widehat{\widetilde{R}}_{\infty,\infty}$ is a $(pg)^{1/p^\infty}$-almost Cohen-Macaulay algebra with respect to $p, x_2, \ldots, x_d$.*
4. *Moreover, if R is a normal domain, there exists a complete unramified regular local ring A together with an integral extension $A \to \widetilde{R}_{\infty,\infty}$ and a nonzero element $h \in A$ such that $A[1/h] \to \widetilde{R}_{\infty,\infty}[1/h]$ is a filtered colimit of finite étale $A[1/h]$-algebras contained in $\widetilde{R}_{\infty,\infty}[1/h]$.*

This can be considered as a generalization of what is constructed for regular local rings as in [Shi16]. In this talk, I will introduce how to prove that the explicitly constructed ring $\widehat{\widetilde{R}}_{\infty,\infty}$ is a perfectoid almost Cohen-Macaulay algebra.

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BIG COHEN-MACAULAY TEST IDEALS IN EQUAL CHARACTERISTIC ZERO VIA ULTRAPRODUCTS

TATSUKI YAMAGUCHI

A (balanced) big Cohen-Macaulay algebra over a Noetherian local ring $(R, \mathfrak{m})$ is an R-algebra B such that every system of parameters is a regular sequence on B. Recently, using big Cohen-Macaulay algebras, Ma and Schwede [2], [3] introduced the notion of BCM test ideals as an analogue of test ideals in tight closure theory. In positive characteristic, Ma-Schwede's BCM test ideals are the same as the generalized test ideals. We consider BCM test ideals in equal characteristic zero.

Using ultraproducts, Schoutens [4] gave an explicit construction of a big Cohen-Macaulay algebra $\mathcal{B}(R)$ over a local domain R essentially of finite type over $\mathbb{C}$. Our main result is stated as follows:

Theorem 1. *Let R be a normal local domain essentially of finite type over $\mathbb{C}$. Let Δ be an effective $\mathbb{Q}$-Weil divisor on* $\operatorname{Spec} R$ *such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, where K_R is a canonical divisor on* $\operatorname{Spec} R$*. Suppose that $\widehat{R}$ and $\widehat{\mathcal{B}(R)}$ are the $\mathfrak{m}$-adic completions of R and $\mathcal{B}(R)$, and $\widehat{\Delta}$ is the flat pullback of Δ by the canonical morphism* $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$*. Then we have*

$$\tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta}) = \mathcal{J}(\widehat{R}, \widehat{\Delta}),$$

where $\tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta})$ is the BCM test ideal of $(\widehat{R}, \widehat{\Delta})$ with respect to $\widehat{\mathcal{B}(R)}$ and $\mathcal{J}(\widehat{R}, \widehat{\Delta})$ is the multiplier ideal of $(\widehat{R}, \widehat{\Delta})$.

As an application of Theorem 1, we show the next result about a behavior of multiplier ideals under pure ring extensions, which is a generalization of [5, Corollary 5.30].

Theorem 2. *Let $R \hookrightarrow S$ be a pure local homomorphism of normal local domains essentially of finite type over $\mathbb{C}$. Suppose that R is $\mathbb{Q}$-Gorenstein. Let Δ_S be an effective $\mathbb{Q}$-Weil divisor such that $K_S + \Delta_S$ is $\mathbb{Q}$-Cartier, where K_S is a canonical divisor on* $\operatorname{Spec} S$*. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $t > 0$ a positive rational number. Then we have*

$$\mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t) \cap R \subseteq \mathcal{J}(R, \mathfrak{a}^t).$$

We discuss a question, a variant of [1, Question 2.7] and consider the equivalence of BCM-rationality and being rational singularities. We also refer to other problems concerning big Cohen-Macaulay algebras in equal characteristic zero.

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GENERAL HYPERPLANE SECTION OF LOG CANONICAL THREEFOLDS IN POSITIVE CHARACTERISTIC

KENTA SATO

First, we recall the definition of klt and lc singularities, which plays an important role in the minimal model program. fs

Definition 1. Let $(R, \mathfrak{m})$ be a normal local ring, essentially of finite type over an algebraically closed field k. Assume that there exists a resolution of singularities $f : Y \to X = \operatorname{Spec} R$ such that the exceptional locus is a simple normal crossing divisor. (This assumption holds if $\mathrm{ch}(k) = 0$ or $\dim R \leqslant 3$.) We say that X has only *klt singularities* (resp. *lc singularities*) if the following two conditions hold:

(1) X is $\mathbb{Q}$-Gorenstein, that is, a canonical divisor K_X of X is $\mathbb{Q}$-Cartier.
(2) Every coefficient of the $\mathbb{Q}$-Weil divisor $K_{Y/X} := K_Y - f^*K_X$ is larger than -1 (resp. larger than or equal to -1).

In this abstract, we work with the following setting.

Setting 2. Let $X \subseteq \mathbb{P}^N_k$ be a normal projective variety over an algebraically closed field k of positive characteristic. We further assume that $\dim X = 3$.

We discuss the following problem

Problem 3. *With notation as in Setting 2, assume that X is klt (resp. lc), that is, the local ring $\mathcal{O}_{X,x}$ is klt (resp. lc) for every point $x \in X$. Then is a general hyperplane section $X \cap H$ klt (resp. lc)? Here, $H \subseteq \mathbb{P}^N_k$ is a hyperplane which is sufficiently general.*

When $\mathrm{ch}(k) > 5$, the klt case of Problem 3 was settled in [ST20].

Fact 4 ([ST20])**.** *With notation as in Setting 2, we further assume that* $\mathrm{ch}(k) > 5$. *If X is klt, then so is a general hyperplane section $X \cap H$.*

In this talk, we consider the lc case.

Theorem 5. *With notation as in Setting 2, we further assume that* $\mathrm{ch}(k) > 3$. *If X is lc, then so is a general hyperplane section $X \cap H$.*

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RINGS OF NILPOTENT ELEMENTS FOR DERIVATIONS IN POLYNOMIAL RINGS

KYOHEI HATTORI AND HIDEO KOJIMA

Let $k[\mathbf{x}] = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k of characteristic zero and let D be a k-derivation of $k[\mathbf{x}]$. Given $b \in k[\mathbf{x}]$, D is *nilpotent at* b if and only if there exists a positive integer m with $D^m(b) = 0$. The set of all elements of $k[\mathbf{x}]$ at which D is nilpotent is denoted by $\mathrm{Nil}(D)$ (cf. [1, p. 3]). It is clear that $\mathrm{Nil}(D)$ is a k-subalgebra of $k[\mathbf{x}]$ and $D|_{\mathrm{Nil}(D)}$ is a locally nilpotent k-derivation on $\mathrm{Nil}(D)$. Although the notion $\mathrm{Nil}(D)$ is well-known for specialists, only a few results are known. Kuroda [3] gave k-derivations Δ on $k[x_1, x_2, x_3, x_4]$ such that $\mathrm{Nil}(\Delta)$ is not finitely generated over k. By using the derivations, he constructed non-locally nilpotent k-derivations on $k[x_1, x_2, x_3, x_4, x_5]$ with slices whose kernels are not finitely generated over k.

In this talk, we give $\mathrm{Nil}(D)$ for some k-derivations D on $k[\mathbf{x}]$. The results of this talk are as follows:

Proposition 1. *Let D be a k-derivation on $k[\mathbf{x}]$ such that $\mathrm{tr.deg}_k \mathrm{Nil}(D) \leq 1$. Then there exists $h \in \mathrm{Nil}(D)$ such that $\mathrm{Nil}(D) = k[h]$.*

Theorem 2. *Let D be a non-zero k-derivation on the polynomial ring $k[x, y]$ in two variables. Assume that $D(x)$ and $D(y)$ are monomial and D is none of the following (1)–(4):*

(1) $y^s\partial_x$ *or* $x^s\partial_y$, *where* $s \in \mathbb{Z}_{\geq 0}$.
(2) $a\partial_x + bx^my^{n+1}\partial_y$ *or* $ax^{m+1}y^n\partial_x + b\partial_y$, *where* $m, n \in \mathbb{Z}_{\geq 0}$ *and* $a, b \in k \setminus \{0\}$.
(3) $ay^m\partial_x + bx^n\partial_y$, *where* $m, n \in \mathbb{Z}_{\geq 0}$ *with* $mn = 0$ *and* $a, b \in k \setminus \{0\}$.
(4) $x^sy^t(nx\partial_x - my\partial_y)$, *where* m *and* n *are relatively prime positive integers with* $mt \neq ns$.

Then $\mathrm{Nil}(D) = \mathrm{Ker}(D)$. *If D is the one as in* (1) *or* (3) *(resp.* (2), (4)*), then* $\mathrm{Nil}(D) = k[x, y]$ *(resp.* $\mathrm{Nil}(D) = k[x]$ *or* $k[y]$, $\mathrm{Nil}(D) = k[x^iy^j \mid (i, j) \in A]$, *where* $A := \{(i, j) \in (\mathbb{Z}_{\geq 0})^2 \mid \exists p \in \mathbb{Z}_{\geq 0}, \exists q \in \mathbb{Z}_{>0} \text{ s.t. } (i, j) + p(s, t) = q(m, n)\}$*).*

The kernels of the monomial derivations on $k[x_1, x_2]$ are determined in [2].

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THE EXISTENCE OF BALANCED NEIGHBORLY POLYNOMIALS

Nguyen Thi Thanh Tam, Hung Vuong University

Joint work with Satoshi Murai

Abstract. Inspired by the definition of balanced neighborly spheres, we define balanced neighborly polynomials and study the existence of these polynomials. The goal of this article is to construct balanced neighborly polynomials of type (k, k, k, k) over any field K for all $k \neq 2$, and show that a balanced neighborly polynomial of type $(2, 2, 2, 2)$ exists if and only if $\mathrm{char}(K) \neq 2$. Besides, we also discuss a relation between balanced neighborly polynomials and balanced neighborly simplicial spheres.

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Gröbner bases of radical Li-Li type ideals

Xin Ren and Kohji Yangawa (Kansai University)

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K. For a partition λ of n and a tableaux T of shape λ, let f_T be the Specht polynomial of T. For example, if $T = \begin{array}{|c|c|c|c|}\hline 4&3&1&7\\ \hline 5&2 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$, then $f_T = (x_4 - x_5)(x_4 - x_6)(x_5 - x_6)(x_3 - x_2)$. The *Specht ideal* I_λ is the ideal of S generated by all f_T for tableaux T of shape λ. Haiman-Woo [1] showed that I_λ is always a radical ideal, and gave a universal Gröbner bases. (This is an unpublished result, and Murai-Ohsugi-Yanagawa [3] gave a quick proof recently.)

On the other hand, an earlier paper of Li-Li [2] studied a class of ideals which is more or less related to Specht ideals. In this talk, we define a class of ideals which generalizes both Specht ideals and *radical* Li-Li ideals (their ideals are not radical in general), and give "Murai-Ohsugi-Yanagawa type results" for this class.

Fix a positive integer l. Let $\lambda = (\lambda_1, \cdots, \lambda_p)$ be a partition of $n + l - 1$ with $\lambda_1 \geq l$. Let $\mathrm{Tab}(l, \lambda)$ be the set of tableaux of shape λ whose letter set is the multiset $\{\overbrace{1, \dots, 1}^{l\text{-copies}}, 2, \dots, n\}$. (For our purpose, to define $\mathrm{Tab}(l, \lambda)$, we may assume that 1's are located in the left squares of the first row like $\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&3&2\\ \hline 4&5&8 \\ \cline{1-3} 6&7 \\ \cline{1-2}\end{array}$.) Consider the ideal $I_{l,\lambda} := (f_T \mid T \in \mathrm{Tab}(l, \lambda)) \subset S$. (Clearly, $I_{1,\lambda} = I_\lambda$). In the sequel, we use a monomial order with $x_n > x_{n-1} > \cdots > x_1$. (Since $I_{l,\lambda}$ is *not* symmetric, giving a *universal* Gröbner bases is difficult.)

Theorem 1. *$I_{l,\lambda}$ is a radical ideal of codimension $\lambda_1 - l + 1$. Moreover, $\{ f_T \mid T \in \mathrm{Tab}(l, \mu), \mu \trianglelefteq \lambda \}$ is a Gröbner bases of this ideal, where $\trianglelefteq$ means the dominance order on partitions of $n + l - 1$.*

Take m with $m \leq n$, and let Δ_m denote the difference product of $x_1, \dots, x_m$. For $T \in \mathrm{Tab}(l, \lambda)$, set $f_{m,T} := \mathrm{lcm}\{f_T, \Delta_m\}$, and consider the ideal $I_{l,m,\lambda} := (f_{m,T} \mid T \in \mathrm{Tab}(l, \lambda)) \subset S$.

Theorem 2. *(While $I_{l,m,\lambda}$ is not radical in general) we have $\sqrt{I_{l,m,\lambda}} = \sum_{\mu \trianglelefteq \lambda} I_{l,m,\mu}$, and $\{ f_{m,T} \mid T \in \mathrm{Tab}(l, \mu), \mu \trianglelefteq \lambda \}$ forms a Gröbner bases of this ideal.*

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WHEN ARE THE NATURAL EMBEDDINGS OF CLASSICAL INVARIANT RINGS PURE?

ANURAG K. SINGH

Abstract: Consider a reductive linear algebraic group G acting linearly on a polynomial ring S over an infinite field; key examples are the general linear group, the symplectic group, the orthogonal group, and the special linear group, with the classical representations as in Weyl's book: for the general linear group, consider a direct sum of copies of the standard representation and copies of the dual; in the other cases take copies of the standard representation. The invariant rings in the respective cases are determinantal rings, rings defined by Pfaffians of alternating matrices, symmetric determinantal rings, and the Plücker coordinate rings of Grassmannians; these are the classical invariant rings of the title, with $S^G \subseteq S$ being the natural embedding.

Over a field of characteristic zero, a reductive group is linearly reductive, and it follows that the invariant ring S^G is a pure subring of S, equivalently, S^G is a direct summand of S as an S^G-module. Over fields of positive characteristic, reductive groups are typically no longer linearly reductive. We determine, in the positive characteristic case, precisely when the inclusion $S^G \subseteq S$ is pure. This is joint work with Melvin Hochster, Jack Jeffries, and Vaibhav Pandey.

GORENSTEIN INDICES OF INVARIANT RINGS

KOHSUKE SHIBATA

This is joint work with Yosuke Nakamura (Tokyo University). Let k be an algebraically closed field of characteristic zero. For a finite group $G \subset \mathrm{GL}_n(k)$ and $g \in G$, we define $d(G)$ and $\mathrm{age}'(g)$ as follows.

Definition 1. Let n be a positive integer and let $G \subset \mathrm{GL}_n(k)$ be a finite subgroup. Let $d := \#G$ be the order of G, and let $\xi \in k$ be a primitive d-th root of unity.

(1) We define a positive integer $d(G)$ by
$$d(G) := \min\{\ell \in \mathbb{Z}_{>0} \mid (\det(g))^{\ell} = 1 \text{ holds for any } g \in G\}.$$

(2) Let $g \in G$. Since g has finite order, g is conjugate to a diagonal matrix $\mathrm{diag}(\xi^{e_1}, \ldots, \xi^{e_n})$ with $1 \leq e_i \leq d$. Then, we define $\mathrm{age}'(g) := \sum_{i=1}^{n} \frac{e_i}{d}$.

The Gorenstein index of a ring R is the order $[\omega_R]$ in $\mathrm{Cl}(R)$. A subgroup of $\mathrm{GL}_n(k)$ is small if it contains no pseudo-reflections. Weston in [Wes91] proved that the Gorenstein index of $k[x_1, \ldots, x_n]^G$ coincides with $d(G)$ when G is small.

The minimal log discrepancy is an invariant of singularities defined in birational geometry. We can see that the minimal log discrepancy of $\mathrm{Spec}\, k[x_1, \ldots, x_n]^G$ at the origin p can be described by $\mathrm{age}'(g)$.

Proposition 2. *Let n be a positive integer and let $G \subset \mathrm{GL}_n(k)$ be a small finite subgroup. Let $p \in \mathrm{Spec}\, k[x_1, \ldots, x_n]^G$ be the origin, i.e. p is the image of the origin of $\mathrm{Spec}\, k[x_1, \ldots, x_n]$. Then it follows that*
$$\mathrm{mld}_p\big(\mathrm{Spec}\, k[x_1, \ldots, x_n]^G\big) = \min\{\mathrm{age}'(g) \mid g \in G\}.$$

Shokurov conjectured that the Gorenstein index of a $\mathbb{Q}$-Gorenstein germ can be bounded in terms of its dimension and minimal log discrepancy.

Conjecture 3 (Shokurov)**.** For any $n \in \mathbb{Z}_{>0}$ and $a \in \mathbb{R}_{\geq 0}$, there exists a positive integer $r(n, a)$ with the following condition.

- If an n-dimensional $\mathbb{Q}$-Gorenstein variety X and a closed point $p \in X$ satisfy $\mathrm{mld}_p(X) = a$, then the Cartier index of K_X at p is at most $r(n, a)$.

The main results in this talk are the following theorem. We prove that Conjecture 3 for quotient singularities.

Theorem 4. *For any $n \in \mathbb{Z}_{>0}$ and $a \in \mathbb{R}_{\geq 0}$, there exists a positive integer $r(n, a)$ with the following condition.*

- *If a small finite subgroup $G \subset \mathrm{GL}_n(k)$ satisfies $\min\{\mathrm{age}'(g) \mid g \in G\} = a$, then the Gorenstein index of $k[x_1, \ldots, x_n]^G$ is at most $r(n, a)$.*

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DEFINING IDEALS OF AFFINE MONOMIAL CURVES IN $\mathbb{A}^4$ AND ASSOCIATED PROJECTIVE MONOMIAL CURVES IN $\mathbb{P}^4$

KAZUFUMI ETO, NAOYUKI MATSUOKA, TAKAHIRO NUMATA, AND KEI-ICHI WATANABE

Let $n_1 < n_2 < \cdots < n_e$ be positive integers with $\gcd(n_1, n_2, \ldots, n_e) = 1$. Let $H = \langle n_1, n_2, \ldots, n_e \rangle$ be the numerical semigroup generated by $n_1, n_2, \ldots, n_e$, that is

$$H = \left\{ \sum_{i=1}^{e} \lambda_i n_i \;\middle|\; 0 \leq \lambda_i \in \mathbb{Z} \right\}$$

We define two semigroups from H.

(1) The dual of H : $H^* = \langle n_e - n_{e-1}, n_e - n_{e-2}, \ldots, n_e - n_1, n_e \rangle$.

(2) The projective closure of H : $\overline{H} = \left\langle \binom{0}{n_e}, \binom{n_1}{n_e - n_1}, \binom{n_2}{n_e - n_2}, \ldots, \binom{n_e}{0} \right\rangle$.

In this talk, we will consider the connection between the structures of $k[H]$, $k[H^*]$, and $k[\overline{H}]$ mainly in the case where $e = 4$.

Date:

TEST ELEMENTS FOR STRETCHEDNESS IN NUMERICAL SEMIGROUP RINGS

MASATAKE IKUMA

This is a joint work with Naoyuki Matsuoka. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. J. Sally introduced the notion of stretched local rings as follows:

Definition 1 ([2])**.** (1) Suppose A is Artinian. We say that A is a stretched Artinian local ring if $\mu_A(\mathfrak{m}^2) \leq 1$.
(2) We say that A is a stretched local ring if there is a parameter ideal Q of A such that Q is a minimal reduction of $\mathfrak{m}$ and A/Q is a stretched Artinian local ring.

In this talk, we will explore a problem when a numerical semigroup ring is stretched.

Let $n > 0$ and $a_1, a_2, \ldots, a_n > 0$ be integers such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. We put

$$H = \langle a_1, a_2, \ldots, a_n \rangle = \left\{ \sum_{i=1}^{n} \lambda_i a_i \;\middle|\; 0 \leq \lambda_i \in \mathbb{Z} \right\}$$

the numerical semigroup ring generated by $a_1, a_2, \ldots, a_n$. Let $S = k[[X_1, X_2, \ldots, X_n]]$ and $V = k[[t]]$ be the formal power series ring over a field k. We consider a ring homomorphism

$$\varphi_H : S \to V$$

defined by $\varphi_H(X_i) = t^{a_i}$ for all $1 \leq i \leq n$. We put $k[[H]] = \operatorname{Im} \varphi_H$ and call it the numerical semigroup ring of H. The defining ideal $\operatorname{Ker} \varphi_H$ of $k[[H]]$ is denoted by I_H.

K. Eto, N. Matsuoka, T. Numata, and K.-i. Watanabe defined the stretchedness for numerical semigroups in their paper [1] in preparation.

Definition 2 ([1])**.** We say that H is stretched if $k[[H]]$ is a stretched local ring.

This definition naturally leads us to expect that it is equivalent to $k[[H]]/(t^e)$ is stretched where e is the multiplicity of $k[[H]]$. However, we have the following.

Example 3 ([1])**.** *Let $H = \langle 6, 7, 11, 15 \rangle$. Then the multiplicity of $k[[H]]$ is 6 and $k[[H]]/(t^6 - t^7)$ is stretched. Since $(t^6 - t^7)$ is a minimal reduction of the maximal ideal $\mathfrak{m} = (t^6, t^7, t^{11}, t^{15})$ of $k[[H]]$, $k[[H]]$ is stretched. Hence H is stretched. But it is easy to check that $k[[H]]/(t^6)$ is not stretched.*

Eto-Matsuoka-Numata-Watanabe stated the following conjecture in [1].

Conjecture 4 ([1])**.** *H is stretched if and only if $k[[H]]/(t^{a_1} - t^{a_2})$ is a stretched Artinian local ring, here we assume $a_1 < a_2 < a_3, \ldots, a_n$.*

In this talk, we will see this conjecture has an affirmative answer when the defining ideal I_H of $k[[H]]$ is generated by 2-minors of a $2 \times n$ matrix with elements in monomials of $X_1, X_2, \ldots, X_n$, namely, after taking a suitable permutation of $a_1, a_2, \ldots, a_n$,

$$I_H = \mathrm{I}_2 \begin{pmatrix} X_2^{m_2} & X_3^{m_3} & \cdots & X_n^{m_n} & X_1^{m_1} \\ X_1^{\ell_1} & X_2^{\ell_2} & \cdots & X_{n-1}^{\ell_{n-1}} & X_n^{\ell_n} \end{pmatrix}$$

for some positive integers $m_1, m_2, \ldots, m_n, \ell_1, \ell_2, \ldots, \ell_n$.

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Dimitrov–Haiden–Katzarkov–Kontsevich complexities for singularity categories

Ryo Takahashi

In 2014, Dimitrov, Haiden, Katzarkov and Kontsevich [2] introduced the notions of complexities and entropies for a triangulated category. In less than a decade since then, a lot of works on these notions have been done; see [3, 4, 5, 6, 7, 8, 9, 12] for instance. Let us recall the definitions.

Definition 1 (Dimitrov–Haiden–Katzarkov–Kontsevich)**.** Let $\mathcal{T}$ be a triangulated category.

(1) Let $A, B \in \mathcal{T}$ and $t \in \mathbb{R}$. We denote by $\delta_t(A, B)$ the infimum of the sums $\sum_{i=1}^{r} e^{n_i t}$, where r runs through the nonnegative integers, and $n_1, \dots, n_r$ run through the integers such that there exists a series

$$\{ B_{i-1} \to B_i \to A[n_i] \rightsquigarrow \}_{i=1}^{r}$$

of exact triangles in $\mathcal{T}$ with $B_0 = 0$ and B_r containing B as a direct summand. The function

$$\mathbb{R} \ni t \mapsto \delta_t(A, B) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is called the *(Dimitrov–Haiden–Katzarkov–Kontsevich) complexity* of B relative to A.

(2) Let $F : \mathcal{T} \to \mathcal{T}$ be an exact functor and $t \in \mathbb{R}$. The *entropy* $\mathrm{h}_t(F)$ of F is defined by

$$\mathrm{h}_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, F^n(G)),$$

where G is a *split generator* of $\mathcal{T}$, i.e., G is an object of $\mathcal{T}$ whose thick closure coincides with $\mathcal{T}$.

Let R be a commutative noetherian local ring. Let $\mathsf{D_{sg}}(R)$ be the *singularity category* of R, which is a triangulated category introduced by Buchweitz [1] and Orlov [10]. In this talk, we explore complexities for $\mathsf{D_{sg}}(R)$. More specifically, we shall consider the following question.

Question 2. Let G be a split generator of $\mathsf{D_{sg}}(R)$. Then does it hold that

$$\delta_t(G, X) = 0$$

for all $X \in \mathsf{D_{sg}}(R)$ and $t \neq 0$?

The contents of this talk will basically come from [11].

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SPECTRA OF DERIVED CATEGORIES OF ALGEBRAIC VARIETIES AND RECONSTRUCTION

HIROKI MATSUI

For a noetherian scheme X, $\mathrm{D}^{\mathrm{pf}}(X)$ is the derived category of perfect complexes on X. In this talk, we consider the following question.

Question 0.1. Let X and Y be noetherian schemes. If $\mathrm{D}^{\mathrm{pf}}(X)$ and $\mathrm{D}^{\mathrm{pf}}(Y)$ are equivalent as triangulated categories, are X and Y isomorphic?

Balmer ([1]) proved that if the equivalence between $\mathrm{D}^{\mathrm{pf}}(X)$ and $\mathrm{D}^{\mathrm{pf}}(Y)$ preserves tensor products, then X and Y are isomorphic as schemes. Indeed, X is obtained as the Balmer spectrum of the tensor triangulated category $\mathrm{D}^{\mathrm{pf}}(X)$: $X \cong \mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X))$. If the equivalence does not preserve tensor products, then it is known that this question is false in general; see [5]. On the contrary, Bondal, Orlov, and Ballard proves the following result:

Theorem 0.2 ([2, 3]). *Let X and Y be Gorenstein projective varieties over a field k. Assume that X and Y have ample or anti-ample canonical bundles. If $\mathrm{D}^{\mathrm{pf}}(X)$ and $\mathrm{D}^{\mathrm{pf}}(Y)$ are k-equivalent as triangulated categories, then X and Y are isomorphic.*

To deal with such a reconstruction, I have introduced the notion of the spectrum of a triangulated category.

Definition 0.3. ([4]) Let $\mathcal{T}$ be a triangulated category. We say that a thick subcategory $\mathcal{P}$ of $\mathcal{T}$ is *prime* if the set

$$\{\mathcal{X} \subseteq \mathcal{T} \mid \mathcal{X} \text{ is a thick subcategory and } \mathcal{P} \subsetneq \mathcal{X}\}$$

has a unique maximal element.

We define the *spectrum* $\mathrm{Spec}_{\triangle}(\mathcal{T})$ of $\mathcal{T}$ as the set of prime thick subcategories of $\mathcal{T}$ together with a certain topology.

The aim of this talk is to give an alternative and algebraic proof of Theorem 0.2 using spectra of derived categories. The key ingredient is the following result:

Theorem 0.4 ([4]). *Let X be a noetherian scheme and let $\mathcal{P}$ be a thick ideal of $\mathrm{D}^{\mathrm{pf}}(X)$. Then*

$$\mathrm{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(X)) \cap \{\textit{thick ideals of } \mathrm{D}^{\mathrm{pf}}(X)\} = \mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X)) \cong X$$

holds.

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REFLEXIVE MODULES OVER THE ENDOMORPHISM ALGEBRAS OF REFLEXIVE TRACE IDEALS

NAOKI ENDO AND SHIRO GOTO

This talk aims at investigating the category of finitely generated reflexive modules over the endomorphism algebras $\mathrm{End}(I) \cong I : I$ of regular reflexive trace ideals I in one-dimensional generically Gorenstein Cohen-Macaulay local rings. The main result generalizes both of the results of S. Goto, N. Matsuoka, and T. T. Phuong ([2, Theorem 5.1]) and T. Kobayashi ([1, Theorem 1.3]) regarding the Gorensteinness of the endomorphism algebra of the maximal ideal. We also explore the question of when the base ring has only finitely many isomorphism classes of indecomposable reflexive modules. We will show that the finiteness of the isomorphism classes implies the ring is analytically unramified and has only finitely many Ulrich ideals. As an application, for example, there are only finitely many Ulrich ideals are contained in Arf local rings once the normalization is finite and is a local ring.

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REALIZING STABLE CATEGORIES OF COHEN-MACAULAY MODULES AS CLUSTER CATEGORIES

OSAMU IYAMA

This talk is based on a joint work with Norihiro Hanihara [HI].

Quiver representations and Cohen-Macaulay representations are two of the main subjects in representation theory of orders. The classical theorems of Gabriel [ASS] and Buchweitz-Greuel-Schreyer [LW, Yo] assert that the class of representation-finite quivers and that of representation-finite Gorenstein rings are parametrized by ADE Dynkin diagrams. Moreover, if R is a simple surface singularity of Dynkin type Q, then the Auslander-Reiten quiver of the derived category $\mathscr{D}^b(\mathsf{mod}\, kQ)$ gives a $\mathbb{Z}$-covering of that of the stable category $\underline{\mathsf{CM}}\, R$ of Cohen-Macaulay R-modules. A theoretical explanation of this observation is given by a triangle equivalence

$$\underline{\mathsf{CM}}\, R \simeq \mathscr{C}_1(kQ), \tag{0.1}$$

where $\mathscr{C}_1(kQ)$ is the 1-cluster category $\mathscr{D}^b(\mathsf{mod}\, kQ)/\tau$ of kQ (see [AIR, Remark 5.9]).

The main aim of this talk is to establish a general theory to construct triangle equivalences between stable categories of large class of Gorenstein rings and cluster categories of finite dimensional algebras.

The category $\mathsf{CM}\, R$ of Cohen-Macaulay modules over a Gorenstein ring R forms a Frobenius category, and its stable category $\underline{\mathsf{CM}}\, R$ has a canonical structure of a triangulated category. By [B], it is triangle equivalent to the *singularity category*:

$$\underline{\mathsf{CM}}\, R \simeq \mathscr{D}_{\mathrm{sg}}(R) := \mathscr{D}^b(\mathsf{mod}\, R)/\,\mathsf{per}\, R. \tag{0.2}$$

It is also classical in Auslander-Reiten theory that, if R is a local isolated singularity of dimension d, then $\mathscr{D}_{\mathrm{sg}}(R)$ is a $(d-1)$-Calabi-Yau triangulated category [A].

On the other hand, cluster categories are Calabi-Yau triangulated categories introduced in this century. The first motivation was to categorify cluster algebras of Fomin-Zelevinsky, and special objects called cluster tilting objects in a cluster category correspond bijectively with clusters in the corresponding cluster algebra. Given a finite dimensional algebra A, its *n-cluster category* $\mathscr{C}_n(A)$ is defined as the triangulated hull of the orbit category $\mathsf{per}\, A/\nu_n$ for the autoequivalence $\nu_n := -\otimes^{\mathrm{L}}_A DA[-n]$.

One of the main tools toward our aim is tilting theory on $\mathbb{Z}$-graded singularity categories. For a $\mathbb{Z}$-graded Gorenstein ring R, we have the $\mathbb{Z}$-graded singularity category $\mathscr{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) := \mathscr{D}^b(\mathsf{mod}^{\mathbb{Z}} R)/\,\mathsf{per}^{\mathbb{Z}} R$, which is equivalent to the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}} R$ of $\mathbb{Z}$-graded Cohen-Macaulay R-modules. Tilting theory enables us to control derived equivalences of rings. More generally, if an algebraic triangulated category $\mathscr{T}$ has a tilting object U, then $\mathscr{T}$ is triangle equivalent to the perfect derived category of $\mathrm{End}_{\mathscr{T}}(U)$. For example, for a simple surface singularity R of Dynkin type Q, there exists a triangle equivalence

$$\mathscr{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) \simeq \mathsf{per}\, kQ,$$

which is a $\mathbb{Z}$-graded version of (0.1) (see [I, Section 5.1][KST]). Tilting theory of $\mathbb{Z}$-graded singularity categories is an active subject in various branches of mathematics including representation theory, commutative algebra, algebraic geometry and mathematical physics, see e.g. [AIR, BIY, HIMO, IT, KST] and a survey article [I].

If R is a $\mathbb{Z}$-graded Gorenstein ring with dimension d and Gorenstein parameter p and there exists a tilting object $U \in \mathscr{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)$ with $A := \mathrm{End}_{\mathscr{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)}(U)$, then by comparing Serre functors, we have a commutative diagram of equivalences

$$\begin{tikzcd}
\mathsf{per}\, A \arrow[r] \arrow[d, dash, "\wr"] & \mathsf{per}\, A/\nu_{d-1} \arrow[r, hook] \arrow[d, dash, "\wr"] & \mathscr{C}_{d-1}(A) \\
\mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}}(R) \arrow[r] & \mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}}(R)/(p) \arrow[r, hook] & \mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}/p\mathbb{Z}}(R).
\end{tikzcd}$$

Since the right inclusions are natural triangulated hulls, one would naively expect an equivalence $\mathscr{C}_{d-1}(A) \simeq \mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}/p\mathbb{Z}}(R)$ on the triangulated hulls. This is far from being obvious since these triangulated hulls are defined using (a priori) different dg enhancements of both categories and functors. Therefore this was shown only in some special cases on a case-by-case basis [AIR, KR, KMV].

Our first main result below justifies the naive expectation above in large generality, and also gives a realization of $\mathscr{D}_{\mathrm{sg}}(R)$ at the same time. For simplicity, here we state it in the easiest form.

Theorem 0.1. *Let $R = \bigoplus_{i\geq 0} R_i$ be a positively graded Gorenstein isolated singularity of dimension $d \geq 0$ with $R_0 = k$ and Gorenstein parameter $p \neq 0$. Suppose $\mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}}(R)$ has a tilting object M. Then $A := \mathrm{End}_{\mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}}(R)}(M)$ is a finite dimensional Iwanaga-Gorenstein algebra, and there is a commutative diagram*

$$\begin{tikzcd}
\mathsf{per}\, A \arrow[r] \arrow[d, dash, "\wr"] & \mathscr{C}_{d-1}(A) \arrow[r] \arrow[d, dash, "\wr"] & \mathscr{C}_{d-1}^{(1/p)}(A) \arrow[d, dash, "\wr"] \\
\mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}}(R) \arrow[r] & \mathscr{D}_{\mathrm{sg}}^{\mathbb{Z}/p\mathbb{Z}}(R) \arrow[r] & \mathscr{D}_{\mathrm{sg}}(R).
\end{tikzcd}$$

Here the category $\mathscr{C}_{d-1}^{(1/p)}(A)$ is the triangulated hull $\mathscr{C}_{d-1}^{(1/p)}(A)$ of the orbit category of $\mathsf{per}\, A$ modulo a p-th root of ν_{d-1}.

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COHEN-MACAULAY RINGS OF HEREDITARY REPRESENTATION TYPE

NORIHIRO HANIHARA

Representation type is a classical notion in representation theory of rings, which measures the complexity of the category one is interested in. We propose to study commutative Cohen-Macaulay rings of "hereditary representation type", by which we mean rings whose representation theory is controlled by finite dimensional hereditary algebras, or in other words, by quiver representations. One can see that Gorenstein rings of finite representation type are of hereditary representation type, and besides finite representation type, one can regard hereditary types as being next simple, from the point of view of representation theory of finite dimensional algebras.

In the talk we would like to present some examples of such rings, by means of tilting theory and cluster tilting theory.

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Cohen-Macaulay representations over Artin-Schelter Gorenstein algebras of dimension one

Yuta Kimura

Representation theory of a Cohen-Macaulay ring A becomes a rich theory if we consider maximal Cohen-Macaulay modules (CM modules for short). We denote by $\mathsf{CM}A$ the category of CM modules. By many results on $\mathsf{CM}A$, such as study of almost split sequences by Auslander-Reiten and Yoshino, the structure of the category is gradually becoming clearer. If A is Gorenstein, then the situation is much nicer. In fact, $\mathsf{CM}A$ becomes a Frobenius category, so the stable category $\underline{\mathsf{CM}}A$ is a triangulated category, and this is equivalent to the singularity category of A [1].

Tilting theory is a powerful tool to study triangulated categories. For a triangulated category $\mathcal{T}$ with middle assumptions, we have a triangle equivalence $\mathcal{T} \simeq \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ for some ring Λ if and only if there exists a tilting object $T \in \mathcal{T}$ satisfying $\mathrm{End}_{\mathcal{T}}(T) \simeq \Lambda$. Thus by finding a tilting object in the stable category of CM modules, we can study the category by using derived categories.

To find a tilting object in the stable category, we assume that A is $\mathbb{N}$-graded. We denote by $\mathsf{CM}^{\mathbb{Z}}A$ the category of $\mathbb{Z}$-graded CM modules. In the case where A has Krull dimension one with middle assumptions, Buchweitz-Iyama-Yamaura [2] characterized when the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}A$ has a tilting object. Namely, there exists a tilting object if and only if the a-invariant of A is non-negative or A is regular. There is a result for existence of tilting objects over quotient singularities [3].

In this talk, we study when the stable category admits a tilting object over an Artin-Schelter Gorenstein algebra (AS Gorenstein for short). AS Gorenstein algebras were introduced as a non-commutative analog of commutative Gorenstein rings from a context of non-commutative algebraic geometry.

The a-invariant of a commutative local Gorenstein ring is given by a grading of an extension between the simple module and the ring. Similarly, for an AS Gorenstein algebra A, a-invariants are defined for each simple modules. We introduce the average a-invariant $a^A_{\mathrm{av}} \in \mathbb{Q}$. To obtain a tilting object, we restrict our category to "locally free on the punctured spectrum" $\mathsf{CM}^{\mathbb{Z}}_0 A$, which is a Frobenius full subcategory of $\mathsf{CM}^{\mathbb{Z}}A$.

Theorem 1. *Let $A = \bigoplus_{i \geq 0} A_i$ be a ring-indecomposable AS Gorenstein algebra of dimension one with the average a-invariant a^A_{av}. Then the triangulated category $\underline{\mathsf{CM}}^{\mathbb{Z}}_0 A$ admits a tilting object if and only if $a^A_{\mathrm{av}} \geq 0$ holds or A has finite global dimension.*

One of main examples of AS Gorenstein algebras is Gorenstein R-orders. Let R be a graded commutative Gorenstein ring. A graded R-algebra A is called Gorenstein R-order if A is a graded CM R-module and $\mathrm{Hom}_R(A, R) \simeq A$ holds as A-modules. Let $R = k[[x]]$ be the ring of formal power series in one variable and $\mathfrak{m}$ the maximal ideal of R. The following matrix shape R-algebras (of sizes 2 and 3) are typical examples of Gorenstein R-orders

$$\begin{pmatrix} R & \mathfrak{m} \\ \mathfrak{m}^a & R \end{pmatrix}, \qquad \begin{pmatrix} R & \mathfrak{m}^a & \mathfrak{m}^b \\ \mathfrak{m}^{-b+c} & R & \mathfrak{m}^c \\ \mathfrak{m}^{a-2b+c} & \mathfrak{m}^{a-b} & R \end{pmatrix}$$

for integers $a, b, c \geq 0$. In the talk, we see the (average) a-invariants of these Gorenstein orders.

This talk is based on joint work with Osamu Iyama and Kenta Ueyama in progress.

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STABLE EQUIVALENCES BETWEEN THE CATEGORIES OF SPHERICAL MODULES AND TORSIONFREE MODULES

YUYA OTAKE

Auslander and Bridger [1] introduced the notion of n-spherical modules for each positive integer n: a finitely generated R-module M is called *n-spherical* if $\mathrm{Ext}_R^i(M,R)=0$ for all $1 \leqslant i \leqslant n-1$ and M has projective dimension at most n. When this is the case, $\mathrm{Ext}_R^i(M,R)=0$ for all $i \neq 0,n$. Auslander and Bridger found various important properties related to n-spherical modules. In this talk, we study the stable category of n-spherical modules. Moreover, we introduce the notion of n-G-spherical modules by replacing projective dimension in the definition of n-spherical modules with Gorenstein dimension, and give similar results for the stable category of n-G-spherical modules. These are related to the category of modules with high grade and the category of totally reflexive modules.

The notion of n-torsionfree modules was also introduced by Auslander and Bridger [1], and played a central role in the stable module theory they developed. The structure of n-torsionfree modules has been well-studied; see [1, 2, 3, 4, 5, 6].

As an application of studies on n-G-spherical modules, we prove that if R is a Gorenstein local ring of Krull dimension $d>0$, then there exists a stable equivalence between the category of $(d-1)$-torsionfree R-modules and the category of d-spherical modules relative to the local cohomology functor.

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ON IDEALS OF INDECOMPOSABLE INTEGRALLY CLOSED MODULES OVER TWO-DIMENSIONAL REGULAR LOCAL RINGS

FUTOSHI HAYASAKA

Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue field. Let M be a finitely generated torsion-free R-module of rank r, and let $F = M^{**}$ be the double R-dual of M. We regard M as a submodule of the R-free module F generated by the columns of a suitable matrix. The ideal of maximal minors of this matrix is denoted by $I(M)$, and we call it the ideal of M. The ideal $I(M)$ is independent of a choice of a matrix, and it is an $\mathfrak{m}$-primary ideal of R if M is not free. Let $\overline{M}$ be the integral closure of M in the sense of Rees [Ree87].

With this notation, Kodiyalam [Kod95] proved the formula $I(\overline{M}) = \overline{I(M)}$ which can be viewed as a generalization of the classical Zariski's product theorem. In particular, the ideal $I(M)$ is an integrally closed ideal if M is integrally closed. Moreover, he proved the following:

Theorem 1 (Kodiyalam)**.** *For any simple integrally closed $\mathfrak{m}$-primary ideal I of R, there exists an indecomposable integrally closed R-module M of rank $r := \mathrm{ord}(I)$ such that $I(M) = I$.*

In particular, there exist indecomposable integrally closed R-modules of arbitrary rank. The modules obtained by this construction are restricted to modules with the simple ideals. The following question was raised in [Kod95]: *Does there exist indecomposable integrally closed R-module M of rank at least two such that the ideal $I(M)$ is non-simple?*

For monomial ideals, a large class of indecomposable integrally closed modules of arbitrary rank with non-simple ideals was constructed in [Hay20, Hay22]:

Theorem 2 (H)**.** *For any (not necessarily simple) integrally closed $\mathfrak{m}$-primary monomial ideal of order $n \geq 2$ satisfying certain conditions, and for any integer r with $2 \leq r \leq n$, there exists an indecomposable integrally closed R-module M of rank r such that $I(M) = I$.*

The modules obtained by this construction are restricted to modules with the monomial ideals. In this talk, we will give a general existence result without this restriction on monomiality. The result is of the following type:

- *Integrally closed ideals satisfying certain conditions occur as the ideal of indecomposable integrally closed modules of rank r.*

On the other hands, it is easy to see that the ideal $\mathfrak{m}^r$ is not the ideal for any indecomposable integrally closed R-modules. This leads to the following problem: *Which ideals can arise as the ideal of an indecomposable integrally closed module?*

In this talk, we will also discuss this problem and give some non-existence results. By putting together these existence and non-existence results, a characterization of ideals that arise as the ideal of an indecomposable integrally closed module will be given in the special cases of rank two and three.

This is joint work with Vijay Kodiyalam.

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ASYMPTOTIC BEHAVIOR OF LOCALIZATIONS OF MODULES

KAITO KIMURA

Throughout this abstract, R is a commutative noetherian ring, I is an ideal of R, and M is a finitely generated R-module. The asymptotic behavior of the quotient modules M/I^nM of M for large integers n is one of the most classical subjects in commutative algebra. Among other things, the asymptotic stability of the associated prime ideals and depths of M/I^nM has been actively studied. Brodmann [1] proved that the set of associated prime ideals of M/I^nM is stable for large n. Kodiyalam [4] showed that the depth of M/I^nM attains a stable constant value for all large n when R is local. There are a lot of studies about this subject; see [2, 5, 6] for instance.

The purpose of this talk is to proceed with the study of the above subject. In particular, we shall consider the following question.

Question 1. Does there exist an integer k such that $\operatorname{depth}(M/I^nM)_{\mathfrak{p}} = \operatorname{depth}(M/I^kM)_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R and all integers $n \geqslant k$?

In this direction, Rotthaus and Şega [6] proved that such an integer k exists if R is excellent, M is Cohen–Macaulay, and I contains an M-regular element. In this talk, we aim to improve their theorem by applying the ideas of their proofs. We give a sufficient condition for the depth of the localization of M/I^nM at any prime ideal of R to be stable for large integers n that do not depend on the prime ideal. One of the main results of this talk gives a common generalization of the above mentioned theorems proved in [4] and [6].

This talk is based on a preprint [3].

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On the Ehrhart ring of the stable set polytope of a cycle graph

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We first fix notation. For sets X and Y, we denote by Y^X the set of maps from X to Y. For a finite set X, we identify $\mathbb{R}^X$ with $\mathbb{R}^{\#X}$, the Euclidean space of dimension $\#X$. For $A \subset X$, we define the characteristic function $\chi_A \in \mathbb{R}^X$ by $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ otherwise. For $f \in \mathbb{R}^X$ and $a \in \mathbb{R}$, we define $af \in \mathbb{R}^X$ by $(af)(x) = a(f(x))$. Further, for a finite set X, $B \subset X$ and $\xi \in \mathbb{R}^X$, we set $\xi^+(B) := \sum_{b \in B} \xi(b)$. We define the empty sum to be 0, i.e., $\xi^+(\emptyset) = 0$.

Let X be a finite set. For a rational convex polytope $\mathscr{P}$ in $\mathbb{R}^X$ and a field $\mathbb{K}$, we define the Ehrhart ring $E_{\mathbb{K}}[\mathscr{P}]$ of $\mathscr{P}$ over $\mathbb{K}$ as follows. Let $-\infty$ be a new element with $-\infty \notin X$. Set $X^- := X \cup \{-\infty\}$. Also, let $\{T_x\}_{x \in X^-}$ be a family of indeterminates indexed by X^-. For $f \in \mathbb{Z}^{X^-}$, we denote the Laurent monomial $\prod_{x \in X^-} T_x^{f(x)}$ by T^f. Then $E_{\mathbb{K}}[\mathscr{P}] := \mathbb{K}[T^f \mid f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f|_X \in \mathscr{P}]$. We set $\deg T_x = 0$ for $x \in X$ and $\deg T_{-\infty} = 1$. Then $E_{\mathbb{K}}[\mathscr{P}]$ is an $\mathbb{N}$-graded subring of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in X^-]$.

In this talk, all graphs are finite simple graphs. A stable set S of a graph $G = (V, E)$ is a subset S of V with no two elements of S are adjacent. The empty set and a set consisting of one vertex is a stable set by the trivial reason.

Definition 1 The steble set polytope $\mathrm{STAB}(G)$ of a graph $G = (V, E)$ is the convex hull of $\{\chi_S \in \mathbb{R}^V \mid S \text{ is a stable set}\}$.

Since $\emptyset$ and $\{v\}$ is a stable set for any $v \in V$, we see that $\dim \mathrm{STAB}(G) = \#V$.

In this talk, we focus our attention to the Ehrhart ring of the stable set polytope of a cycle graph $G = (V, E)$. A cycle graph is a graph consisting of one cycle. It is known that $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is Gorenstein if and only if the length of the cycle is even or less than 6. Therefore, we assume in the following that $\#V = 2\ell + 1$, where ℓ is an integer with $\ell \geq 3$ and set $V = \{v_0, v_1, \ldots, v_{2\ell}\}$, $E = \{\{v_i, v_j\} \mid i - j \equiv 1 \pmod{2\ell + 1}\}$. and $R = E_{\mathbb{K}}[\mathrm{STAB}(G)]$.

For i with $0 \leq i \leq 2\ell$, we set

$$\mathfrak{p}_i := \bigoplus_{\substack{\mu \in \mathbb{Z}^{V^-},\ T^\mu \in E_{\mathbb{K}}[\mathrm{STAB}(G)], \\ \mu(v_i) > 0 \text{ or } \mu^+(V) < \ell\mu(-\infty)}} \mathbb{K} T^\mu.$$

Then $\mathfrak{p}_i$ is a prime ideal of R and $\dim R/\mathfrak{p}_i = \ell + 1$.

Theorem 2 *The non-Gorenstein locus of* $\mathrm{Spec} R$ *is* $V(\bigcap_{i=0}^{2\ell} \mathfrak{p}_i)$.

Further, we show the following.

Theorem 3 *R is almost Gorenstein.*

Finally, we show that the conjecture of Hibi and Tsuchiya is true. Let $(h_0, h_1, \ldots, h_s)$, $h_s \neq 0$ be the h-vector of R. Since $\dim R = 2\ell + 2$ and $a(R) = -3$, we see that $s = 2\ell - 1$. Hibi and Tsuchiya conjectured the following.

Conjecture 4 *$h_{s-t} = h_t$ for $0 \leq t \leq 1$ and $h_{s-t} = h_t + (-1)^t$ for $2 \leq t \leq \ell - 1$.*

Theorem 5 *Conjecture 4 is true.*